Nonlinear Robust Disturbance Rejection

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Declaration

These doctoral studies were conducted with Professor Brian Anderson, Dr Thomas Brinsmead, and Dr Matthew James as supervisors. The contents of this thesis are the results of original research carried out by myself, in collaboration with others, and have not been submitted for a higher degree at any other university or institution.

Much of the work in this dissertation has been published, has been submitted or is close to be submitted for publication in book chapters, refereed journals and conference proceedings. In some cases, the conference papers contain material overlapping with the journal publications. Where I am first author of the work, I contributed at least 50%.

Book Chapters

- S.W. Su, B.D.O. Anderson, and T.S. Brinsmead, "Constant disturbance rejection and zero steady state tracking error for nonlinear systems design", In Biswa Datta, editor, *Applied computational control, signals, and circuits 2001*, Pages 1-30, KLUWER, BOSTON.

Journal Papers

- S.W. Su, B.D.O. Anderson, and T.S. Brinsmead, "Use of integrator in nonlinear $\mathcal{H}_\infty$ design for disturbance rejection", *Automatica*. To be published.
- B.D.O. Anderson, S.W. Su, and T.S. Brinsmead, "The minimal generic multi-realisation of linear multi-variable systems in the areas of multiple model adaptive control", *Automatica*. To be submitted.
- S.W. Su, B.D.O. Anderson, and T.S. Brinsmead, "The minimal stably based multi-realisation of linear multi-variable systems in the areas of multiple model adaptive control", *IEEE Transaction on Automatic Control*. To be submitted.

Conference Proceedings

- S.W. Su, B.D.O. Anderson, and T.S. Brinsmead, "Robust disturbance suppression for nonlinear systems based on $\mathcal{H}_\infty$ control", in *Proceedings of the IEEE Conference on Decision and Control*, 39th, December, 2000, Pages 3013-3018, Sydney, Australia.
- S.W. Su, B.D.O. Anderson, and T.S. Brinsmead, "Constant disturbance suppression for nonlinear systems design using singular perturbation theory", in *Proceedings of the IEEE Conference on Decision and Control*, 40th, December, 2001, Pages 3272-3277, Orlando, Florida, USA.

- S.W. Su, B.D.O. Anderson, and T.S. Brinsmead, "Robust disturbance suppression for nonlinear systems based on multiple model adaptive control", Submitted to *Proceedings of the IEEE Conference on Decision and Control*, 41th, 2002, USA.

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Abstract

This thesis studies the problem of robust disturbance rejection for nonlinear systems based on three different methods: $\mathcal{H}_\infty$ control, singular perturbation theory and multiple model adaptive control.

Firstly, the disturbance suppression problem for nonlinear systems based on $\mathcal{H}_\infty$ control is examined. We review the so-called nonstandard mixed sensitivity problem, which introduces an integrator into a selected weight, as well as the linear classical disturbance suppression problem and the linear $\mathcal{H}_\infty$ disturbance suppression problem. We extend this $\mathcal{H}_\infty$ problem to the nonlinear case, and present a method to reduce the order of the state feedback Hamilton-Jacobi PDE (Partial Differential Equation) for this nonlinear $\mathcal{H}_\infty$ problem by extending the concept of comprehensive stability. Finally, we investigate the structure of the output feedback $\mathcal{H}_\infty$ controller for disturbance suppression, and draw the conclusion that, as in the linear case, there must also be an integrator in the controller.

Secondly, a relatively practical method of suppressing the effect of constant disturbances on nonlinear systems is presented. By adding an integrator to a stabilising controller, it is possible to achieve both constant disturbance rejection and zero tracking error. Sufficient conditions for the rejection of a constant input disturbance are given. We give both local and global conditions such that the inclusion of an integrator in the closed loop maintains closed loop stability. The analysis is based on singular perturbation theory. Furthermore, we present some alternative locations for adding an integrator into the closed loop system and extend these methods to deal with Multiple-input Multiple-output nonlinear systems. Finally, we implement our method in the control of a simulated helicopter model. The simulation results show that this method achieves satisfactory performance.

In the last part of this thesis, we apply multiple model adaptive control to deal with the robust disturbance rejection problem for an unknown plant. Firstly, a stable multi-estimator for an unstable nonlinear plant is constructed, based on the concept of a stable kernel representation. An example is presented demonstrating the design of a multi-estimator and a multi-controller to ensure constant disturbance rejection as well as constant reference tracking under plant variation. The simulation results indicate that satisfactory performance is achieved. Finally, an efficient way to achieve

a multi-realisation for multi-controller and multi-estimator structures, named minimal (and minimal generic) stably based feedback multi-realisation, is presented for linear multi-variable systems. Although we have not presented a comprehensive theory for multi-controllers and multi-estimators for nonlinear systems (in contrast to an example demonstrating their feasibility), we have constructed part of the basis of such a theory, in the consideration of MMAC for MIMO linear time invariant systems.

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Notation

$\mathcal{R}$	field of real numbers
$\mathcal{C}$	field of complex numbers
$j\mathcal{R}$	imaginary axis
$\in$	belong to
$\subset$	subset
$\cap$	intersection
$\cup$	union
$\ll$	much less than
$\gg$	much greater than
α^*	complex conjugate of $\alpha \in \mathcal{C}$
$Re\{\alpha\}$	real part of $\alpha \in \mathcal{C}$
I_n	$n \times n$ identity matrix
$[a_{ij}]$	a matrix with a_{ij} as its i-th row and j-th column element
$diag\{a_1, \cdots, a_n\}$	an $n \times n$ matrix with a_i as its i-th diagonal element
A^T	transpose
A^*	adjoint operator of A or complex conjugate transpose of A
A^{-1}	inverse of A
$det\{A\}$	determinant of A
$\lambda\{A\}$	eigenvalue of A
$\bar{\sigma}\{A\}$	largest singular value of A
$\|\cdot\|$	norm (normally Euclidean norm $\|x\| := \sqrt{\langle x, x\rangle}$)
$\|\cdot\|_p$	p–norm, e.g. $(\sum_i \|x_i\|^p)^{1/p}$
$\|A\|$	operator norm of A, i.e. $\sup_x \|Ax\|/\|x\|$
$\mathcal{R}[s]$	ring of polynomials in s with real coefficient
$\mathcal{R}(s)$	the field of fractions associated with $\mathcal{R}[s]$

Chapter 1

Introduction

The topic of this thesis is **nonlinear robust disturbance rejection**. The adjective **nonlinear** is to be interpreted as **not necessarily linear**. The reason for investigating nonlinear systems is the fact that virtually **all physical systems** are nonlinear in nature [79]. There exist sound techniques that enable us to approximate a physical system as a linear model (a set of ordinary linear differential equations); for instance, one can linearise a nonlinear system around various operation points, if this approximation does not deviate too much from the underlying physical system. Therefore, the analysis of linear systems occupies an important place in system theory. However, in analysing the behaviour of any physical system, it is often the case that the linearised model is not adequate or accurate enough. This forces us to consider nonlinear models of physical plants.

Because linear systems form only a small sub-set of nonlinear systems, research on nonlinear systems is significantly different from that on linear systems. Firstly, it is often possible to derive closed-form expressions for the solutions of the system equation for linear systems. In general, this is not possible for nonlinear systems. For nonlinear systems qualitative analysis may have to be used to make some predictions about the behaviour of a nonlinear system. Secondly, the analysis of nonlinear systems makes use of a wider variety of approaches and mathematical tools than does the analysis of linear systems. Thirdly, many results for nonlinear systems only provide sufficient conditions (rather than necessary and sufficient conditions for linear systems).

The adjective **robust** in the thesis title can be interpreted more generally as the ability for systems to "work well" under the system's uncertainty and environment disturbances. There are mainly two reasons for research on robust control systems. One is that uncertainty is inevitable. Physical systems are essentially infinite dimensional, time-varying and nonlinear; many are better modelled by nonlinear partial differential equations. Uncertainty arises because any mathematical model being used for design purposes can do no more than approximate the behaviour of the real system that is to be controlled [80]. The other main reason is that there will always be uncertainty about the actual numerial values of the various parameters of the model. The aim for control system design is to ensure the controlled systems possess robust stability and robust performance. One methodology for robust control design is the $\mathcal{H}_\infty$ design technique, which can achieve robust stability and robust performance under unstructed uncertainty and external disturbances. Adaptive control that mainly deals with parameter uncertainty and external disturbances is also regarded as a robust control design method in this thesis.

Disturbance rejection can be interpreted as minimising the effects of external disturbances by control system design. The problem of disturbance rejection (especially constant disturbance rejection) arises in many industrial fields, such as motion-control, active noise control and vibration control. In this thesis, the constant disturbance (a form of unstable disturbance) rejection problem is comprehensively investigated.

1.1 Background

In this thesis, robust disturbance suppression for a nonlinear system is achieved by feedback controller design. The standard feedback configuration is shown in Figure 1.1.

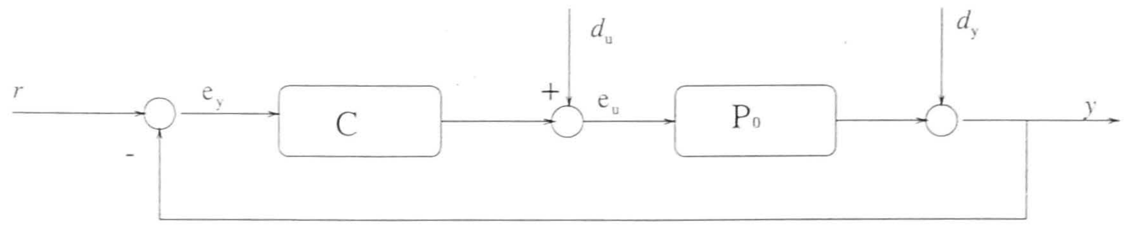

Figure 1.1: A classical disturbance suppression problem

It consists of the interconnection of a plant P_0 and controller C forced by a command signal r, as well as an input disturbance d_u and an output disturbance d_y. The plant P_0 may be nonlinear and with some uncertainty. Our aim is to achieve reference tracking and disturbance rejection for the nonlinear uncertainty plant by design of the controller C.

This thesis deals with the robust disturbance rejection problem for nonlinear systems using three methods respectively: $\mathcal{H}_\infty$ control design, singular perturbation theory and multiple model adaptive control.

$\mathcal{H}_\infty$ Control

The mathematical symbol $\mathcal{H}_\infty$ stands for the Hardy space [32] [1] of all complex-valued functions of a complex variable, which are analytic and bounded in the open right-half complex plane. For a linear (continuous-time, time-invariant) plant, the $\mathcal{H}_\infty$ norm of a transfer matrix is the maximum of its largest singular value over all frequencies [14].

The "standard" linear $\mathcal{H}_\infty$ control problem is concerned with the block diagram in Figure 1.2.

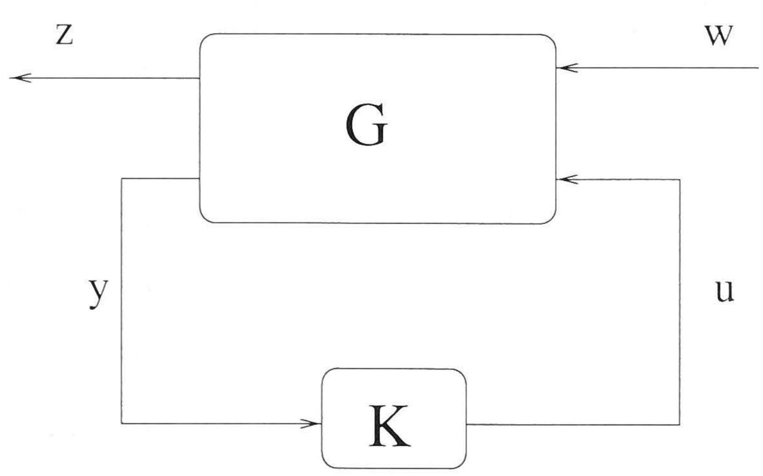

Figure 1.2: A "standard" linear $\mathcal{H}_\infty$ control problem.

In Figure 1.2, w represents an external disturbance, y is the measurement available to the controller, u is the output from the controller, and z is an error signal that it is desired to keep small. The transfer function matrix G incorporates not only the conventional plant to be controlled but also any weighting functions included to specify the desired performance. The linear $\mathcal{H}_\infty$ control problem is then to design a stabilising controller K, so as to ensure the $\mathcal{H}_\infty$ norm of the closed loop transfer function T_{wz} from w to z satisfying $\|T_{wz}\|_\infty \leq \gamma$, where

$$\|T_{wz}\|_\infty = \sup_\omega \bar{\sigma}(T_{wz}(j\omega)). \tag{1.1}$$

The $\mathcal{H}_\infty$ norm gives the maximum energy gain. An important property of the $\mathcal{H}_\infty$ norm is evident in its application with the small gain theorem, which states that if

[1]The $\mathcal{H}_2$ and $\mathcal{H}_\infty$ space with the $\mathcal{H}_p$ space, $p \geq 1$, are usually called Hardy spaces.

$\|T_{wz}\|_\infty \leq \gamma$ then the system with a block diagram as follows will be stable for all stable Δ with $\|\Delta\|_\infty < 1/\gamma$ [84].

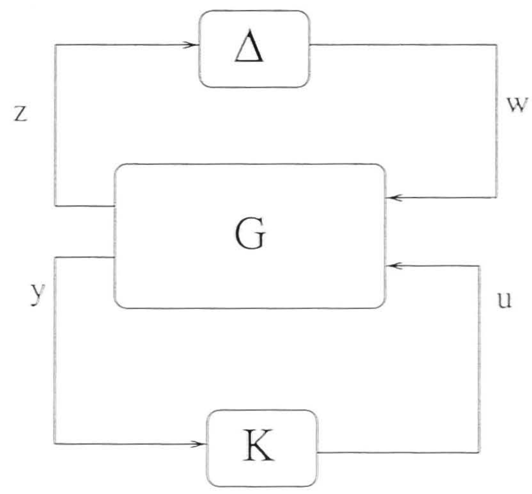

Figure 1.3: An $\mathcal{H}_\infty$ control system under bounded stable uncertainty Δ.

Controller design problems where the $\mathcal{H}_\infty$ norm plays an important role were initially formulated by George Zames in the early 1980s [83], in the context of sensitivity reduction in linear plants, with the design problem posed as a mathematical optimisation problem using the $\mathcal{H}_\infty$ operator norm. They were formulated originally in an input-output setting (in the frequency domain) and involved analytic functions or operator-theoretic methods. The main tools used during the early phases of research on this class of problems have been spectral factorisation, (Youla) parametrisation, and operator and approximation theory. However, the first solution to a general rational MIMO $\mathcal{H}_\infty$ optimal control problem, presented in [21], relied heavily on state-space methods.

In the late 1980s, Glover et al. [25] and Doyle et al. [22] developed the so-called two Algebraic Riccati Equation(ARE) algorithm as a solution to the "standard" linear $\mathcal{H}_\infty$ control problem. Relations between $\mathcal{H}_\infty$ control have been established with many other topics in control, such as risk sensitivity control, differential games, J-lossless factorisation, maximum entropy methods, and so on [84].

The standard $\mathcal{H}_\infty$ control algorithm requires several pre-requisite conditions, which are often violated in industrial applications. Problems where the conditions are not fulfilled are called nonstandard $\mathcal{H}_\infty$ control problems. The standard $\mathcal{H}_\infty$ control algorithm will be introduced as follows to highlight the difference between standard and nonstandard $\mathcal{H}_\infty$ control problems.

Consider a generalised linear plant with its stabilisable and detectable realisation described as

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} = \begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{1.2}$$

where $z \in \mathcal{R}^m$, $y \in \mathcal{R}^q$, $w \in \mathcal{R}^r$ and $u \in \mathcal{R}^p$ are the controlled output, the measurement output, the disturbance input and the control input, respectively. The problem is to find a proper control law $u(s) = K(s)y(s)$ that internally stabilises the closed loop system and satisfies $\|T_{wz}(s)\|_\infty < 1$, where $T_{wz}(s)$ is the closed loop transfer function from w to z given by the following lower linear fractional transformation(LFT):

$$T_{wz}(s) = F_l(P; K) := [P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}](s). \tag{1.3}$$

The word standard connotes that the plant (1.2) satisfies the following assumptions:

(1) (A, B_2, C_2) is stabilisable and detectable.

(2) $rank\{D_{12}\} = p$, $rank\{D_{21}\} = q$.

(3)

$$rank \begin{bmatrix} -j\omega I + A & B_2 \\ C_1 & D_{12} \end{bmatrix} = n + p, \quad rank \begin{bmatrix} -j\omega I + A & B_1 \\ C_2 & D_{21} \end{bmatrix} = n + q, \forall \omega.$$

The assumption (1) is necessary for there to exist a stabilising controller $K(s)$. The assumptions (2) and (3) mean that $P_{12}(s)$ and $P_{21}(s)$, with their realisations induced from (1.2), do not have invariant zeros on the $j\omega$-axis including infinity. An $\mathcal{H}_\infty$ control problem is called nonstandard or singular if one or more of the conditions (1), (2) and (3) do/does not hold.

The study of robust nonlinear control has attracted increasing interest over the last few decades [11] [12] [13] [14] [20] [27] [28] [34] [36] [45]. The nonlinear $\mathcal{H}_\infty$ control methodology is intended to be a means of designing controllers for nonlinear systems. The aim of controller design is to make nonlinear closed loop systems that are internally stable and have an induced $\mathcal{L}_2$ gain less than a specified number. Progress has been aided by the recent extension of the linear quadratic results which link the theories of $\mathcal{L}_2$ gain control (nonlinear $\mathcal{H}_\infty$ control), differential games, and stochastic risk sensitive control [28].

Now, let us go back to the robust disturbance rejection problems for linear system first. In recent years, $\mathcal{H}_\infty$ methods have been employed to handle disturbance suppression problems [49] [84] for linear systems. The main methodological device is to introduce an integrator in a selected weight function and then formulate the disturbance rejection problem as a mixed sensitivity problem. Here, the mixed sensitivity problem is the problem of simultaneously achieving bounds on weighted versions of the sensitivity and complementary sensitivity functions (See [49]). However, these problems are nonstandard $\mathcal{H}_\infty$ problems, because they have an un-stabilisable pole at the origin, which violates the pre-requisite conditions of standard $\mathcal{H}_\infty$ control theory. There are several indirect ways to get around this problem, such as by using singular perturbation techniques or changing the system block diagram to absorb the integrator weight into the control loop [84].

Paper [49] uses so-called extended $\mathcal{H}_\infty$ theory to give a relatively direct alternative solution of this nonstandard $\mathcal{H}_\infty$ problem for linear systems. Furthermore, the integrator weighting leads to order reduction of the Riccati equation by using a so-called quasi-stabilising solution. As for a classical control design, the controller arising from either of the two $\mathcal{H}_\infty$ approaches in [49] and [84] normally contains an integrator.

In this thesis, we extend these ideas to the nonlinear disturbance rejection problem. As in the linear case, for the general nonlinear $\mathcal{H}_\infty$ problem it is convenient to regard some problems as standard [49] [50] [84], the remaining ones then being nonstandard. Many papers and books [20] [34] [28] on nonlinear $\mathcal{H}_\infty$ control deal exclusively with standard nonlinear $\mathcal{H}_\infty$ control problems. In this thesis, we consider issues that arise due to the state-feedback $\mathcal{H}_\infty$ problem being non-standard, assuming that we already have access to a state measurement or estimate. We do not discuss the construction of an appropriate state-estimate. For output feedback problems, there are two broad approaches for constructing an $\mathcal{H}_\infty$ state estimate. In [20] and [34], a finite dimensional filter is constructed leading to local sufficient and global necessary conditions for the existence of an output-feedback controller. In contrast [28] exploits information state ideas, leading to an infinite dimensional filter equation, which nevertheless, leads to necessary and sufficient conditions for solving the $\mathcal{H}_\infty$ output feedback problem. Each of [20] [28] and [34], however, deals with the standard $\mathcal{H}_\infty$ problem. In this paper, we investigate the constant disturbance rejection problem. Not surprisingly, the $\mathcal{H}_\infty$ constant disturbance rejection problem that we consider for the nonlinear case inherits the difficulty of the linear case: the existence of un-stabilisable states makes the problem nonstandard.

Singular Perturbation Theory

Ever since Prandtl's work at the beginning of last century [40], singular perturbation techniques have been a traditional tool of fluid dynamics. The singular perturbation model of finite-dimensional dynamic systems was extensively studied in the mathematical literature from the 1940s to the 1960s. Its use also spread to other areas of mathematical physics and engineering.

In the control literature, the singular perturbation approach of [76] and [77] was first applied to optimal control and regulator design by Koktovic and Sannuti [40]. Applications to broader classes of control problems followed at an increasing rate.

For control engineers, singular perturbations are a means of taking into account neglected high-frequency phenomena and considering them in a separate fast time-scale [43]. This is achieved by treating a change in the dynamic order of a system of dynamic equations as a parameter perturbation, which, being more abrupt than a regular perturbation, is called a singular perturbation. The practical advantages of such a "parameterisation" of changes in model order are significant, because the order of every real dynamic system is higher than that of the model used to represent the system.

The singular perturbation theory is closely connected with composite Lyapunov function design for the recently developed constructive nonlinear control method [42] [60].

A singular perturbation model is [60]

$$\left\{ \begin{array}{lcl} \dot{x} & = & f_c(x, z, u), x \in \mathcal{R}^{n_x} \\ \mu \dot{z} & = & q_c(x, z, u), z \in \mathcal{R}^{n_z} \end{array} \right. \tag{1.4}$$

where $\mu > 0$ is the singular perturbation parameter.

A fundamental property of the singular perturbation model is that it possesses two time scales: the slow time scale of the x-dynamics, and a fast time scale of the z-dynamics. The separation of time scales is parameterised by μ.

It is common practice to neglect the dynamics of the system that are much faster than the rest of the system [60]. In this case we have to deal with fast unmodelled dynamics. The separation of time scales into slow and fast allows the design to be performed on the nominal slow model. This has been proven by the theory of singular perturbation theory [58].

More specifically, when we let the nominal feedback control law be $u = -k(x)$ and denote

$$\begin{aligned} f_c(x, z, -k(x)) &= f(x, z), \\ q_c(x, z, -k(x)) &= q(x, z), \end{aligned} \tag{1.5}$$

we obtain the standard singular perturbation form

$$\dot{x} = f(x, z), x \in \mathcal{R}^{n_x} \tag{1.6}$$

$$\mu \dot{z} = q(x, z), z \in \mathcal{R}^{n_z} \tag{1.7}$$

Without loss of generality, we assume that $f(0, 0) = 0$ and $q(0, 0) = 0$.

Now, let the following assumptions be satisfied:

(i) The equation

$$0 = q(x, z)$$

obtained by setting $\mu = 0$ in equation (1.7) has a unique $\mathcal{C}^2$ solution $z = \bar{z}(x)$.

(ii)For any fixed $x \in \mathcal{R}^{n_x}$ the equilibrium $z_e = \bar{z}(x)$ of the subsystem (1.7) is globally asymptotically stable and locally exponentially stable.

(iii) The equilibrium $x = 0$ of the reduced (slow) model

$$\dot{x} = f(x, \bar{z}(x))$$

is globally asymptotically stable and locally exponentially stable.

Then, for every two compact sets $\mathcal{C}_x \in \mathcal{R}^{n_x}$ and $\mathcal{C}_z \in \mathcal{R}^{n_z}$ there exists $\mu^* > 0$ such that for all $0 < \mu < \mu^*$ the equilibrium $(x, z) = (0, 0)$ of the whole system (1.6 and 1.7) is asymptotically stable and its region of attraction contains $\mathcal{C}_x \times \mathcal{C}_z$.

In this thesis, we address the problem of achieving constant input disturbance rejection and constant reference tracking [73], for nonlinear systems based on singular perturbation theory.

While the constant disturbance suppression and reference tracking problems are reasonably well understood for linear systems, their solution usually requires the inclusion of an error signal integrator in the controller. How to deal with these problems is not so well understood for nonlinear systems. Using singular perturbation methods enables the modification of already existing stabilising controllers to suppress the constant disturbance while still retaining the stability of the system. These ideas are developed in Chapter 3.

Multiple model adaptive control

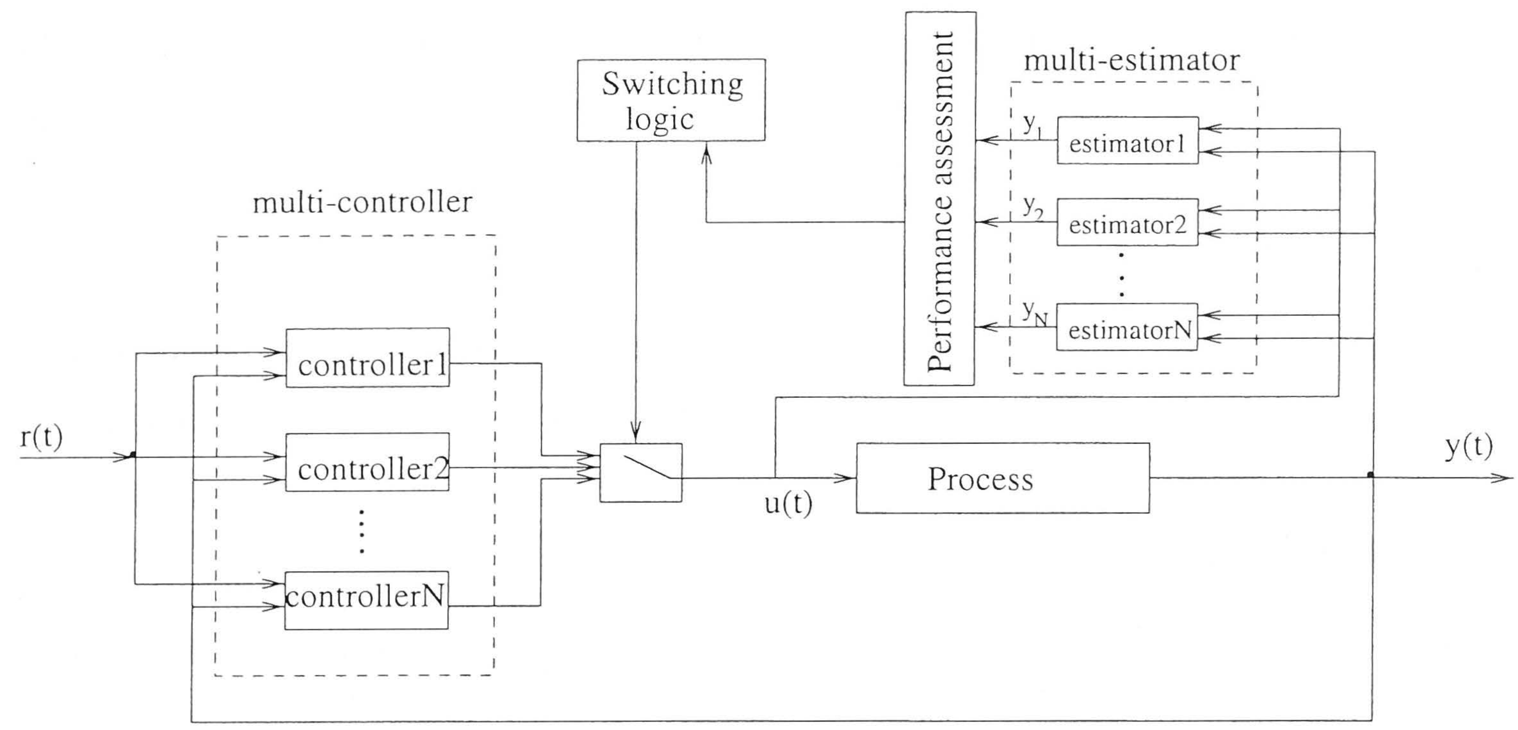

Figure 1.4: A Multiple Model Adaptive Control system.

Multiple model adaptive control(MMAC) is a model-based control strategy that incorporates a set of model/controller pairs rather than relying on a single model and controller to handle all possible operating conditions (see Figure 1.4). More precisely, MMAC algorithms assume that the unknown true plant either belongs to a given finite set of nominal plants, or is at least in some way close to one (or more) members of that set [5]. Each nominal plant corresponds to a controller that is presumed to give satisfactory performance in conjunction with both the nominal plant, and the associated uncertainty ball.

A number of excellent text and monographs ([4], [5], [29], [52], [54] and [57]) have been written in the area of MMAC, especially for linear plants. Paper [54] actually provides a way to achieve robust (constant) disturbance suppression and constant reference tracking for a linear SISO plant based on a supervised control system. The main disturbance suppression methodology, not unsurprisingly, is to integrate the reference tracking error by including an integrator in the controller. It has been shown in [54] that the supervisory part of the controller can orchestrate the switching of a sequence of candidate controllers into feedback with the system so as (i) to cause the output of the process to approach and track a constant reference input despite norm-bounded unmodelled dynamics, and constant process disturbances and (ii) to ensure that none of the signals within the overall system can grow without bound in response to bounded disturbances, be they constant or not.

This thesis is a first step in the direction of extending some of these ideas to nonlinear plants. The key issue is to explore how to achieve an MMAC capability, and this is in turn rests on having a so-called stable multi-estimator. The stable multi-estimator for a possibly unstable nonlinear plant is constructed based on stable kernel representations. This is one way an extension of the linear system ideas in the papers [5] and [53] can be achieved.

Then, an efficient way of multi-realisation for multi-controller and multi-estimator structure, named minimal stably based multi-realisation, is presented for linear multivariable systems.

Just as one can consider a standard linear system realisation problem (given a transfer function, find a state-variable realisation), and a minimal realisation problem (ensure the state-variable realisation is of minimal degree), so for a finite collection of transfer functions can one consider a multi-realisation problem. The transfer functions here are those of the family of controllers or estimators. As argued, in for example [52], because at any instant of time only one of the constituent controller is to be applied to the Plant, it is only necessary to generate one candidate control signal. Often this means significant simplification can be achieved if all control signals are generated by a single system. In other words, rather than implementing each of the controllers in the family as a separate dynamical system, one can often achieve the same results using a single controller with adjustable parameters (see Definition 4.6). Because the single controller state is, in effect, shared by the family of controllers, we call this implementation a state sharing multi-realisation using parameter dependent feedback. A well-known problem in switching control is the poor transient response that can arise due to controller switching. State sharing will ameliorate this kind of problem.

Almost all of the literature on system realisation methods deals with the implementation of a single linear time invariant (LTI) system [7] [17] [37] [51] [81] [82] based on one of a state space description approach, a matrix fraction description approach or a geometric approach. Morse [52] presented some results for the multi-realisation of several linear SISO systems in the context of examining MMAC for scalar plants. In this thesis, we investigate the multi-realisation of several linear MIMO systems.

The results will be applicable to MMAC problems for MIMO plants. This is done as a first step towards a comprehensive theory of multi-controllers and multi-estimators for nonlinear systems.

Why is it a first step? Why should a nonlinear theory, even one for single-input, single-output systems reflect more of the ideas in MIMO linear systems than in SISO linear systems?

The first answer to this question flows from reflecting on the fractional representations of linear systems and the nonlinear generalisations thereof. A SISO rational transfer function $\frac{n(s)}{d(s)}$, where $n(s)$ and $d(s)$ are not polynomial but rather stable proper transfer functions, can be thought of both as nd^{-1} and $d^{-1}n$, and it can commute with any other SISO rational transfer function. These properties are indeed called upon in the MMAC theory set out in for example [52]. SISO nonlinear fractional descriptions (image or kernel representations) do not display the two properties just mentioned, and in this sense are more like linear fractional description of MIMO systems. Accordingly, turning a linear theory (of multi-estimators or multi-controllers) into a nonlinear theory will almost certainly be easier if the linear theory is for MIMO systems.

A second answer to the question can also be advanced. It may well be that nonlinear MMAC can be tackled in the first instance by using linearised models. If a model is a linearisation of a nonlinear system around a trajectory rather than a single operating point, it will be time-varying. Again, the SISO (time-invariant) idea of indistinguishability of left and right factorisations and commutativity, fall away because of the time-variation.

A SISO linear time-varying system in some ways has more in common with MIMO linear time-invariant systems than a SISO linear time-invariant systems.

Thus we contend that, although we have not presented a comprehensive theory for multi-controllers and multi-estimators for nonlinear systems (in contrast to an example demonstrating their feasibility), we have constructed part of the basis of such a theory, in the consideration of MMAC for MIMO linear time invariant systems.

1.2 Contributions of the Thesis

In this thesis, we focus on robust disturbance rejection for nonlinear systems. We present three main contributions related to: nonlinear $\mathcal{H}_\infty$ control, application of singular perturbation theory and multiple model adaptive control.

Robust disturbance rejection for nonlinear systems based on $\mathcal{H}_\infty$ control

As in the linear case, for the nonlinear $\mathcal{H}_\infty$ problem there are once again standard and nonstandard problems. Not surprisingly, the $\mathcal{H}_\infty$ disturbance rejection problem for the nonlinear case inherits the difficulty of the linear case: the existence of unstabilisable states makes the problem nonstandard.

Paper [49] uses so-called *extended $\mathcal{H}_\infty$ theory* to give a relatively direct solution to the disturbance suppression nonstandard $\mathcal{H}_\infty$ problem for linear systems. Furthermore, the integrator weighting leads to order reduction of the Riccati equation by using the concept of a so-called quasi-stabilising solution. As for a classical control design, the controller arising from the $\mathcal{H}_\infty$ approaches normally contains an integrator. We extend these ideas to the nonlinear disturbance suppression problem.

The main bottleneck of nonlinear $\mathcal{H}_\infty$ control, which is similar to the problem encountered in nonlinear optimal control, is the need to solve the Hamilton-Jacobi (HJ) partial differential equation (PDE) [45]. Although explicit globally-defined solutions of most HJ PDEs are hard to obtain, we [72] have presented a method which can simplify (by order reduction) the HJ PDE for the nonlinear disturbance rejection problem by using the concept of *comprehensive stability*, which is extended from the linear case (see [49]).

Next, we research the problem of using output feedback, rather than state feedback, to achieve constant disturbance suppression. We first show how one might use a nonlinear observer in conjunction with a state-feedback $\mathcal{H}_\infty$ controller in order to develop an output feedback controller. The form of solution suggests that an output feedback controller that rejects constant disturbances may contain an integrator. We confirm that the controller normally acquires an integrator, a phenomenon well known in the linear case, and the basis of classical constant disturbance suppression ideas.

Nonlinear $\mathcal{H}_\infty$ output feedback control is particularly difficult. The standard solution of the linear $\mathcal{H}_\infty$ output feedback control problem normally depends upon solving two Riccati equations [84]. One of these, which arises in the state feedback control problem, is replaced by an HJ PDE in the nonlinear case. The other, however, is replaced by a still more complicated equation (involving an information state), see [28]. Practical approaches to the solution of this latter equation are so far lacking.

As mentioned above, to circumvent this difficulty we suppose that for our particular system an observer is constructed to estimate the unknown states. We then substitute the state $\hat{x}$ for the true state x in the state feedback controller, and then check the γ-dissipativity and stability of the resulting closed loop system.

In the linear case, the disturbance rejection output feedback controller necessarily includes an integrator in it unless the plant has a zero at the origin. We have given theorems which show that for the nonlinear case the controller still needs an integrator.

The main contributions on this topic are summarised below.

- We provide an extension of nonlinear $\mathcal{H}_\infty$ theory to deal with constant disturbance rejection for nonlinear systems.
- We achieve an order reduced HJ-PDE for the state feedback nonlinear $\mathcal{H}_\infty$ problem by using the concept of *comprehensive stability*.
- We conclude that for nonlinear output feedback, the controller must contain an integrator to reject constant disturbance.

Robust disturbance rejection for nonlinear systems based on singular perturbation theory

We stated above that for disturbance suppression, an output feedback controller must contain an integrator. In this section we ask whether we can directly add an integrator to an already existing controller to achieve constant disturbance rejection, while still retaining the stability of the system. Often that would be both a simpler and more practical way to deal with the nonlinear constant disturbance suppression problem.

We not only give the affirmative answer but also suggest several locations where an integrator with low gain away from DC ($\frac{\epsilon}{s}$ for short) may be included, in order to deal with the constant input disturbance rejection problem. Furthermore, this method can also be applied to deal with the constant reference tracking problem, even for nonlinear MIMO systems, such as the helicopter system of [41].

We show that if a controller is augmented with an integrator, and the closed loop is exponentially stable, then input-output stability is ensured. Note that there is an integrator weight function which ensures that, even for a constant disturbance, the output signal is in $\mathcal{L}_2$ and hence asymptotically goes to zero. That is, the controller suitably augmented with a low gain integrator $\frac{\epsilon}{s}$ will reject a constant input disturbance.

We present both local and global conditions such that the inclusion of an integrator in the closed loop maintains closed loop stability. The analysis is based on singular perturbation theory. The global condition for the closed loop stability is that the "incremental DC gain" of the nonlinear plant is uniformly bounded away from zero [70], and the local condition for the closed loop stability is just that the local "incremental DC gain" is nonzero.

Next, we give some alternative locations for adding an integrator into the closed loop system and extend these methods to deal with nonlinear MIMO systems.

Finally, we implement our method in the control of a simulated helicopter model (figure 3.10). The simulation results (see figure 3.11) show that this method achieves satisfactory performance [70] [73].

The main contributions on this topic are summarised below.

- We provide a relatively practical method of dealing with a constant disturbance for nonlinear systems: adding $\frac{\epsilon}{s}$ to an existing stabilising controller.

- We present local and global conditions for the existence of ϵ^* to retain the stability of a system augmented with $\frac{\epsilon}{s}$.

- We give a lower bound on the value of ϵ^* based on singular perturbation theory.

- We extend this method to MIMO nonlinear systems, and provide a simple method to design the gain matrix design of the augmented system.

- We design a disturbance rejection controller for a helicopter model. Simulation shows that we achieve a satisfactory disturbance suppression result.

Robust disturbance rejection for nonlinear systems based on multiple model adaptive control

Although MMAC is model-based, it relaxes the requirement for a single and precise model. Often such a model is not available either due to lack of plant knowledge or insufficient time for the development of detailed models.

Paper [54] provided a way to achieve robust (constant) disturbance suppression and constant reference tracking for linear SISO plant based on a supervised control system. The main methodology is to integrate the reference tracking error by including an integrator into the controller, a similar idea to the use of the singular perturbation method as described in the previous section.

The purpose of this part is to modestly extend these ideas to the nonlinear case based on previous results. It is no doubt that a full extension is a huge work. Therefore, we mainly consider two aspects of this problem.

Firstly, a stable multi-estimator for an (open-loop) unstable nonlinear plant is constructed based on the concept of a stable kernel representation. This is a direct extension of papers [53] and [5].

Then, an efficient way of multi-realisation for multi-controller and multi-estimator structure, named minimal stably based feedback multi-realisation, is presented for linear multi-variable systems and a class of nonlinear SISO systems. This is an extension of paper [52], which provides a method of stably based feedback multi-realisation for linear SISO systems.

As mentioned above, although we have not presented a comprehensive theory for multi-controllers and multi-estimators for nonlinear systems (in contrast to an example demonstrating their feasibility), we have constructed part of the basis of such a theory, in the consideration of MMAC for MIMO linear time invariant systems.

The main contributions on this topic are summarised below.

- We present a method for the construction of a multi-estimator for nonlinear systems based on the normalised stable kernel representation. An example for nonlinear MMAC design is provided, and satisfactory simulation results are achieved by using Matlab Simulink.

- We present the results for the multi-realisation of a number of linear SISO systems, and highlight some fundamental issues such as the relationship between feedback multi-realisation and coprime factorisation.

- We provide the necessary and sufficient conditions for the multi-realisation of a family of linear multi-variable systems based on matrix fractional descriptions.

- We introduce the new concept of hc-dependence, and provide the necessary and sufficient conditions for hc-dependence.

- We solve the minimal (and minimal "generic") stably based multi-realisation problems for linear MIMO systems based on hc-dependence.

1.3 Summary of Thesis Contents

Chapter 2 Here we deal with the disturbance suppression problem based on $\mathcal{H}_\infty$ control. In Section 1, we briefly review the classical constant disturbance suppression method. In Section 2, we examine, for linear systems, the mixed sensitivity $\mathcal{H}_\infty$ method, and in particular, the so-called extended $\mathcal{H}_\infty$ method which can deal with the robust constant disturbance suppression problem. In Section 3, we set up the disturbance suppression problem for the nonlinear case. In Section 4, we present the main results for this chapter, giving an order reduction theorem for the state feedback HJ PDE arising from the nonlinear constant disturbance suppression problem. Finally, in Section 5, we probe the structure of the output feedback $\mathcal{H}_\infty$ controller of the system under consideration and show that it normally contains an integrator.

Chapter 3 We address the disturbance rejection problem based on singular perturbation theory. In Section 1, we give a description of the problem. In Section 2, we present a proof that an exponentially stabilising nonlinear controller appropriately augmented with a small integrator (a linear transfer function $\frac{\epsilon}{s}$) can achieve constant disturbance suppression. In Section 3, we give both local and global conditions for the existence of a gain of the integrator that is sufficiently small to guarantee stability. Section 4, by using singular perturbation methods, gives an upper bound on a value of the gain that guarantees closed loop stability. In Section 5, we suggest alternative locations for adding an integrator into the system. Section 6 extends our method to deal with the constant disturbance rejection problem and constant reference tracking problem for MIMO systems. Finally, in Section 7, we present simulation results obtained by implementing constant disturbance rejection and zero steady state tracking error control for a helicopter model by using this method.

Chapter 4 We investigate two aspects of the disturbance suppression problem for the nonlinear system based on multiple model adaptive control. In Section 1, we provide a method for the construction of a multi-estimator for even nonlinear unstable systems based on stable kernel representation. A frame work of disturbance rejection for nonlinear systems based on MMAC is presented by a simulation example. In Section 2, the problems of the minimal (and minimal "generic") stably based multi-realisation for a family of linear multi-variable systems are presented and solved.

A summary of the main results of this thesis and conclusions, appears in the final chapter,**Chapter 5**.

Chapter 2

Robust disturbance rejection for nonlinear systems base on $\mathcal{H}_\infty$ control

The disturbance suppression problem for nonlinear systems is examined in this chapter. We review the so-called nonstandard mixed sensitivity problem, which introduces an integrator to a selected weight, as well as the linear classical disturbance suppression problem and the linear $\mathcal{H}_\infty$ disturbance suppression problem. We extend this $\mathcal{H}_\infty$ problem to the nonlinear case, and present a method to reduce the order of the state feedback Hamilton-Jacobi (HJ) Partial Differential Equation (PDE) for this nonlinear $\mathcal{H}_\infty$ problem by extending the concept of comprehensive stability [50] [49]. Finally, we investigate the structure of the output feedback $\mathcal{H}_\infty$ controller for disturbance suppression, and draw the conclusion that, as in the linear case, there must also be an integrator in the controller.

2.1 Introduction

This chapter is mainly concerned with the constant disturbance rejection problem for nonlinear systems, and it uses $\mathcal{H}_\infty$ methods to examine the problem. An important objective of control system design is to minimise the effects of external disturbances. The problem of disturbance rejection (especially constant disturbance rejection) arises in many industrial fields, such as motion-control, active noise control and vibration control. The classical method for constant disturbance rejection is to include an integrator into the controller. However, the classical disturbance suppression technique requires separate consideration of stability, so it cannot directly deal with the robust stability issue.

In recent years, $\mathcal{H}_\infty$ methods have been employed to handle disturbance suppression problems [84] [49] for linear systems. The main methodological device is to introduce an integrator in a selected weight function and then formulate the disturbance rejection problem as a mixed sensitivity problem. However, these mixed sensitivity problems are nonstandard $\mathcal{H}_\infty$ problems, because they have an un-stabilisable pole at the origin, which violates the pre-requisite conditions of standard $\mathcal{H}_\infty$ control theory. There are several indirect ways to get around this problem, such as by using singular perturbation techniques or by alternatively changing the system to an equivalent block diagram which absorbs the integrator weight into the control loop [84].

Paper [49] uses so-called extended $\mathcal{H}_\infty$ theory to give a relatively direct alternative solution of this nonstandard $\mathcal{H}_\infty$ problem for linear systems. Furthermore, the integrator weighting leads to order reduction of the Riccati equation by using the concept of a so-called quasi-stabilising solution. As for a classical control design, the controller arising from either of the two $\mathcal{H}_\infty$ approaches normally contains an integrator. The purpose of this chapter is to try to carry over these ideas to the nonlinear disturbance suppression problem. As in the linear case, for the nonlinear $\mathcal{H}_\infty$ problem there are once again standard and nonstandard problems. Not surprisingly, the $\mathcal{H}_\infty$ disturbance rejection problem for the nonlinear case inherits the difficulty of the linear case: the existence of un-stabilisable states makes the problem nonstandard.

The main bottleneck of nonlinear $\mathcal{H}_\infty$ control, which is similar to the problem encountered in nonlinear optimal control, is the need to solve the Hamilton-Jacobi (HJ) partial differential equation (PDE)[45]. Although explicit globally-defined solutions of most HJ PDEs are hard to obtain, we will present a method which can simplify (by order reduction) the HJ PDE for the nonlinear disturbance rejection problem by using the concept of *comprehensive stability*, which is extended from the linear case (See [49]). Furthermore, we can show that the controller for output feedback control contains an integrator in a sense defined later, in Section 2.6.

In the next section, we briefly review the classical constant disturbance suppression

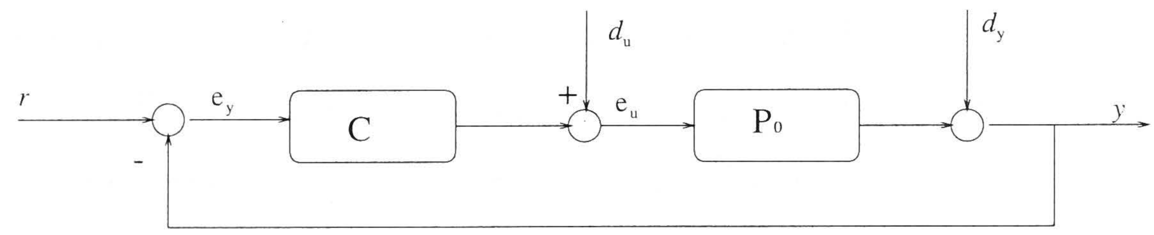

Figure 2.1: A classical disturbance suppression problem

method. In Section 2.3, we examine, for linear systems, the mixed sensitivity $\mathcal{H}_\infty$ method, and in particular, the so-called extended $\mathcal{H}_\infty$ method, which can deal with the robust constant disturbance suppression problem. In Section 2.4, we set up the disturbance suppression problem for the nonlinear case. Section 2.5, the main part, gives an order reduction theorem for the state feedback HJ PDE arising from the nonlinear constant disturbance suppression problem. Finally in Section 2.6, we probe the structure of the output feedback $\mathcal{H}_\infty$ controller of the system under consideration, and show that it normally contains an integrator.

2.2 Review of the classical constant disturbance rejection technique for linear systems

Let us consider a classical disturbance rejection problem as shown in Figure 2.1. This depicts a linear time-invariant single-input single output (SISO) system. It consists of the interconnection of a plant $P_0(s)$ and controller $C(s)$ forced by a command signal r, as well as an input disturbance d_u and an output disturbance d_y. In this chapter, we will pay more attention to the constant input disturbance rejection problem.

From Figure 2.1, we can write the transfer function from r, d_u and d_y to e_y and e_u as follows.

$$\begin{bmatrix} e_u(s) \\ e_y(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{1+P_0(s)C(s)} & \frac{C(s)}{1+C(s)P_0(s)} & \frac{-C(s)}{1+C(s)P_0(s)} \\ \frac{P_0(s)}{1+P_0(s)C(s)} & \frac{1}{1+C(s)P_0(s)} & \frac{-1}{1+C(s)P_0(s)} \end{bmatrix} \begin{bmatrix} d_u(s) \\ r(s) \\ d_y(s) \end{bmatrix} \tag{2.1}$$

This equation can be used to study three different aspects: rejection of input constant disturbances, rejection of output constant disturbances, and tracking of a constant reference signal.

For the input constant disturbance rejection problem, we are normally interested in reducing or eliminating the effect of the disturbance d_u. When d_u changes slowly (i.e. the bandwidth of the input disturbance is low pass), according to classical control the desired effect can be achieved by adding an integrator into the controller C, provided stability is retained.

In the following we will not consider the case that $P_0(0) = 0$, because if this is the case, the input constant disturbance d_u will not influence the output and furthermore reference tracking and output disturbance rejection is impossible.

In more detail, for the transfer function (2.1), we can check that if the command $r(s)$ and output disturbance $d_y(s)$ are identically zero and $d_u(s)$ is a step, i.e. $d_u(s) = 1/s$, then an integral controller (i.e. $C(s) = \hat{C}(s)/s$ with $\hat{C}(0)$ nonzero) can totally reject the disturbance on e_u when $t \to \infty$ (given that $P_0(0)$ is nonzero). It is also necessary that $C(s)$ contains an integrator to secure the rejection property on e_u if P_0 does not contain an integrator.

Similarly, retaining the assumption that $P_0(0) \neq 0$, from the transfer function (2.1), we can see that an integrator in the controller is not only sufficient, but also necessary for ensuring constant input disturbance rejection on e_y (when $t \to \infty$). An integrator in the plant alone will not suffice.

For the constant reference tracking problem and the output disturbance rejection problem, we need to consider the static tracking error $e_y(\infty)$. From the transfer function (2.1), we can check that if the command $r(s)$ or output disturbance $d_y(s)$ is a step, then an integrator in $C(s)P_0(s)$ can secure the static tracking error goes to zero(when $t \to \infty$).

To summarise, $P_0(0) \neq 0$ is necessary for the possibility of constant reference tracking. Under this assumption, desirable properties(disturbance rejection and zero tracking error) follow when P_0C has an integrator, and if d_u is presented and e_y is to go to zero, the integrator must be in C.

2.3 $\mathcal{H}_\infty$ treatment of the classical disturbance suppression problem for linear systems

The classical disturbance suppression technique demands separate theoretical consideration of stability, and certainly does not deal directly with the robust stability issue.

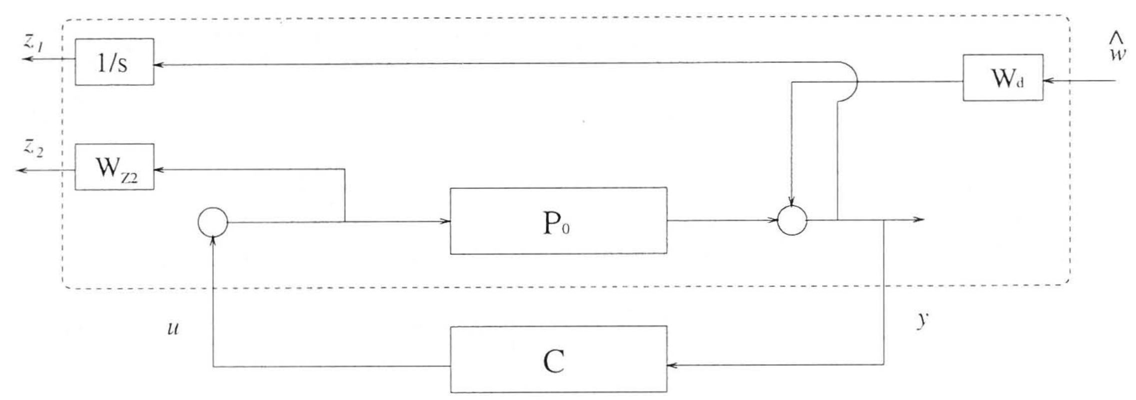

Figure 2.2: The linear $\mathcal{H}_\infty$ framework for the disturbance suppression problem with an integral in the output weight

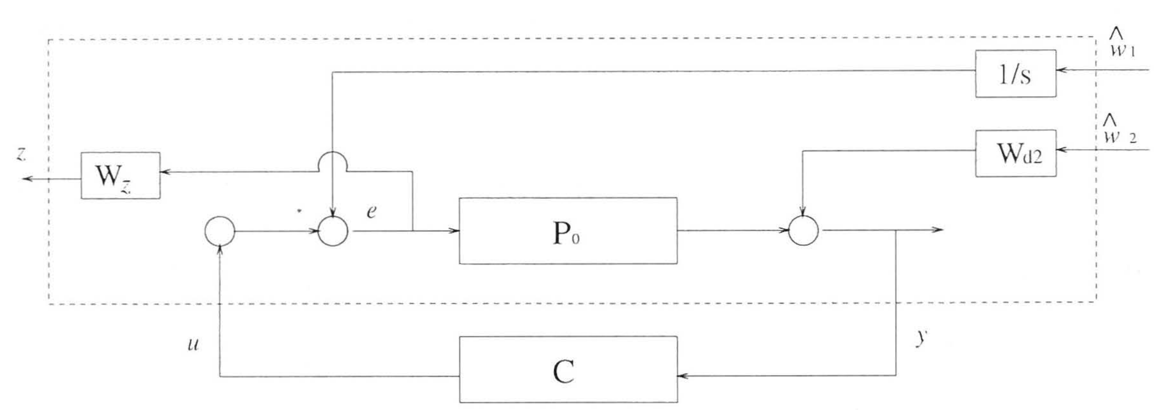

Figure 2.3: The linear $\mathcal{H}_\infty$ framework for the disturbance suppression problem with an integral in the input weight

To guarantee robust stability, we need to rely on a theory of robust control, such as $\mathcal{H}_\infty$ theory. There are at least two ways [84][49] to design an $\mathcal{H}_\infty$ controller for the disturbance suppression problem. Apart from redrawing the loop of Figure 2.1 to correspond to the standard $\mathcal{H}_\infty$ formulation, the main methodological device is to introduce an integrator into a selected weight function. The $\mathcal{H}_\infty$ problems in [84] and [49] belong to the class of mixed sensitivity problems in $\mathcal{H}_\infty$ control. In [84] an integrator is introduced into one of the output weights W_z (see Figure 2.2), while in [49] there is an integrator in one of the input weights W_d (Figure 2.3).

Evidently, associated with the classical disturbance suppression problem (Figure 2.1), there are at least two kinds of mixed sensitivity $\mathcal{H}_\infty$ problems, (Figures 2.2 and 2.3). It can be easily checked, for the linear case, that the two $\mathcal{H}_\infty$ problems are duals of each other. So, without loss of generality, we can choose the mixed sensitivity problem described in Fig 2.3 as the basis for our discussions.

In this diagram, P_0 represents the given plant, $1/s$ and W_{d2} are input weights, W_z is an output weight, and C is the controller which needs to be constructed in such a way that it can stabilise the plant P_0, and make the infinity norm of the transfer function from $[\hat{w}_1\ \hat{w}_2]^T$ to z less than some given bound γ. Note that at zero frequency the integrator ensures that the gain from the integrator output to z will be zero, and this is the mechanism for achieving constant disturbance suppression. Given the plant and the weights, the standard approach is to seek to formulate the problem as an $\mathcal{H}_\infty$ problem. However, this problem does not satisfy all the pre-requisite conditions of the standard $\mathcal{H}_\infty$ control problem (which includes a stability condition [26] [84]), because of an un-stabilisable mode at the origin. Therefore, this problem is termed nonstandard. More precisely, consider the state-variable realisation of the "generalised plant" with input $\hat{w}_1, \hat{w}_2$ and u and output z and y in Figure 2.3. The entire state is not stabilisable from u, because the integrator driven by $\hat{w}_1$ is unaffected by u.

The book [84] gives indirect solutions for such nonstandard problems by using singular perturbation methods (using $\frac{1}{s+\epsilon}$ instead of $\frac{1}{s}$ for small positive ϵ).

The so-called extended $\mathcal{H}_\infty$ controller [49] will solve the mixed sensitivity problem described in Figure 2.3, where a constant disturbance enters at the plant input. By using disturbance-observer-based integral control, the robust stability requirement is satisfied directly. The synthesis of the extended $\mathcal{H}_\infty$ controller requires a "quasi-stabilising" solution [49] of the "X"-Riccati equation (the Riccati equation arising in the state feedback problem, which also arises in the output feedback problem.). The original $(n+1)$-th order Riccati equation can be constructed from the solution of a reduced order n-th order equation, n being the degree of P_0.

Similarly we can use extended $\mathcal{H}_\infty$ controller design to solve the mixed sensitivity problem of Figure 2, where the constant disturbance enters at the plant output. Not surprisingly, for this dual formulation, it is possible to simplify the controller synthesis by constructing the solution to the original $(n+1)$-th order "Y" Riccati equation (the Riccati equation arising in the output feedback routines which is termed the filter or observer Riccati equation) from the solution of a reduced n-th order equation.

2.4 Setting up the disturbance suppression problem formulation in the nonlinear case

The problems discussed above are all linear ones. In this section, we give a description of the nonlinear problem.

To begin, we consider the classical disturbance problem shown as in Figure 2.1, except that the plant may be nonlinear. In order to give a more explicit description, we suppose that the SISO nonlinear plant, P_0, is modelled as follows.

$$P_0 : \left\{ \begin{array}{rcl} \dot{x_0} & = & A(x_0) + B_1(x_0)w_1 + B_2(x_0)u \\ y & = & C_2(x_0) + w_2. \end{array} \right. \tag{2.2}$$

We assume that the functions appearing in systems of this chapter are smooth with bounded first and second order partial derivatives. Here, $w = [w_1\ w_2]^T$, and $w_1 \in R$ corresponds to a plant input disturbance, while $w_2 \in R$ corresponds to a plant output disturbance. The introduction of the disturbance w_2 can be interpreted as a way of capturing modelling uncertainty for output feedback $\mathcal{H}_\infty$ control. It should be noted that if w_2 is zero, then the problem becomes singular. In order to simplify our discussion, we shall suppose that $B_1(x_0) = B_2(x_0)$. (For the input disturbance rejection problem, this condition is always satisfied.)

We need to extend this nonlinear disturbance rejection problem as depicted in Figure 2.1 to an $\mathcal{H}_\infty$ style problem. As mentioned in the previous section, for the linear case [84] [49], there are two ways to perform this step. The first one is depicted in Figure 2.2, and the second one in Figure 2.3. The first way, as stated in the last section, leads to a solution allowing order reduction of the "Y"-Riccati equation (or observer Riccati equation) in the linear case. However, for the nonlinear case, there is no simple and explicit "filter" HJ PDE which is equivalent to the "Y" Riccati equation of linear $\mathcal{H}_\infty$ system theory. If we choose the second formulation, it turns out that we can reduce the order of the control HJ PDE for the state feedback problem (which is equivalent to the "X" Riccati equation in the linear case). Therfore, we elect to extend this nonlinear disturbance suppression problem to an $\mathcal{H}_\infty$ problem along the lines of [49].

The framework of this nonlinear $\mathcal{H}_\infty$ problem is shown in Figure 2.4. In order to slightly extend the application scope of our method, we choose $\frac{1}{s}G_{w1}(s)$ instead of only $\frac{1}{s}$ as the weight of $\hat{w_1}$. Here $G_{w1}(s)$ is a stable and proper transfer function. Based on linear classical control theory, $\frac{1}{s}G_{w1}(s)$ can be written as $\frac{\alpha}{s} + G_{w12}(s)$, where α is real and $G_{w12}(s)$ is a stable and strictly proper transfer function (See Figure 2.5.)

The state equation of the weight transfer function becomes:

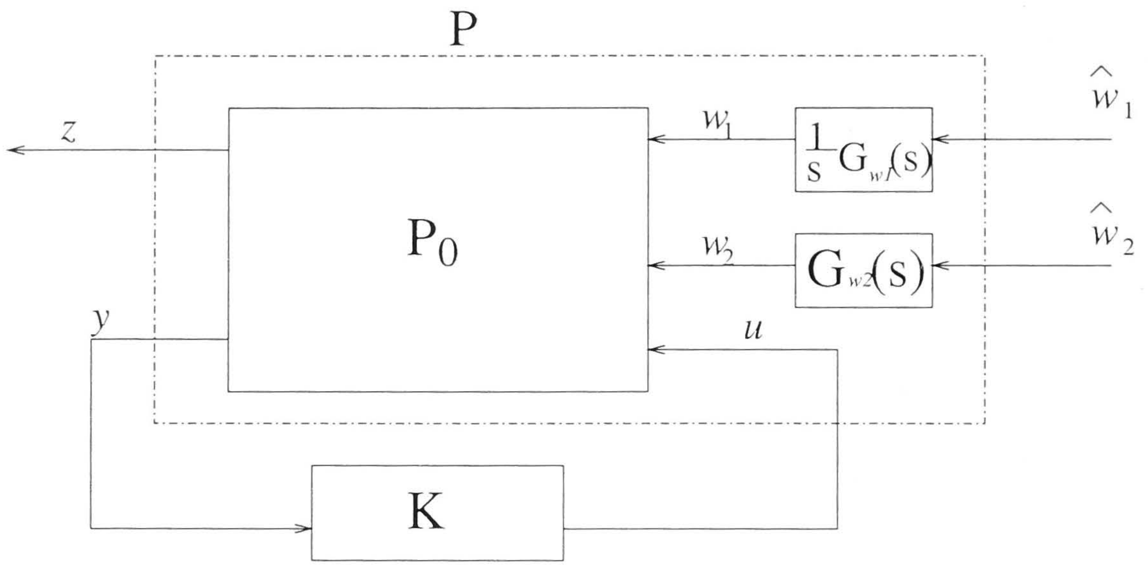

Figure 2.4: A nonlinear disturbance suppression problem

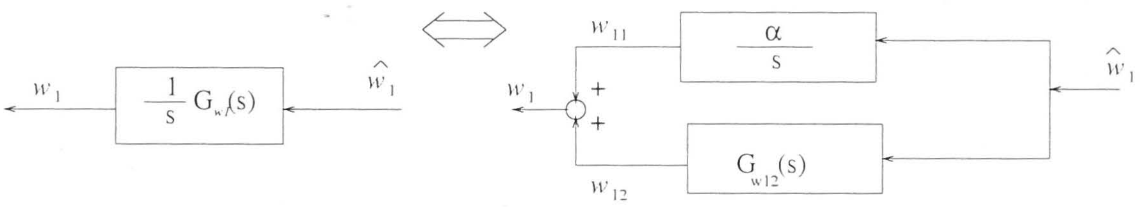

Figure 2.5: The change of weight transfer function

$$\left\{\begin{array}{lcl} \dot{x}_{w_{11}} & = & \alpha\hat{w}_1 \\ \dot{x}_{w_{12}} & = & A_{w_{12}}x_{w_{12}} + B_{w_{12}}\hat{w}_1 \\ w_1 & = & w_{11} + w_{12} = x_{w_{11}} + C_{w_{12}}x_{w_{12}} \end{array}\right. \tag{2.3}$$

We assume that $G_{w_2}(s)$ in Figure 2.4 is a rational stable proper transfer function with no finite and infinite $j\omega$-axis zeros, which can be expressed as:

$$\left\{\begin{array}{lcl} \dot{x}_{w_2} & = & A_{w_2}x_{w_2} + B_{w_2}\hat{w}_2 \\ w_2 & = & C_{w_2}x_{w_2} + D_{w_2}\hat{w}_2 \end{array}\right. \tag{2.4}$$

The equations of the generalised system are now:

$$\left\{\begin{array}{lcl} \dot{x}_{w_{11}} & = & \alpha\hat{w}_1 \\ \dot{x}_{w_{12}} & = & A_{w_{12}}x_{w_{12}} + B_{w_{12}}\hat{w}_1 \\ \dot{x}_{w_2} & = & A_{w_2}x_{w_2} + B_{w_2}\hat{w}_2 \\ \dot{x}_0 & = & A(x_0) + B_1(x_0)(x_{w_{11}} + C_{w_{12}}x_{w_{12}} + u) \\ y & = & C_2(x_0) + C_{w_2}x_{w_2} + D_{w_2}\hat{w}_2 \end{array}\right. \tag{2.5}$$

The choice of z is important. We elect to set $z = e'$ (See Figure 2.6), a small difference from the linear case. [1] That is, we split the disturbance w_1 into two

[1]For the linear case, $z = e = y_{w_1} + u$. We also could choose $z = e$ here, but it is more convenient

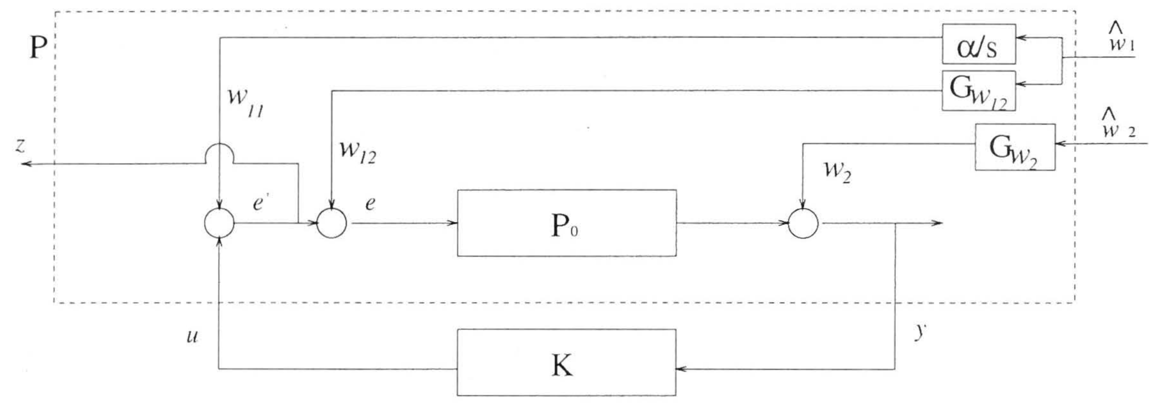

Figure 2.6: The explanation for e'

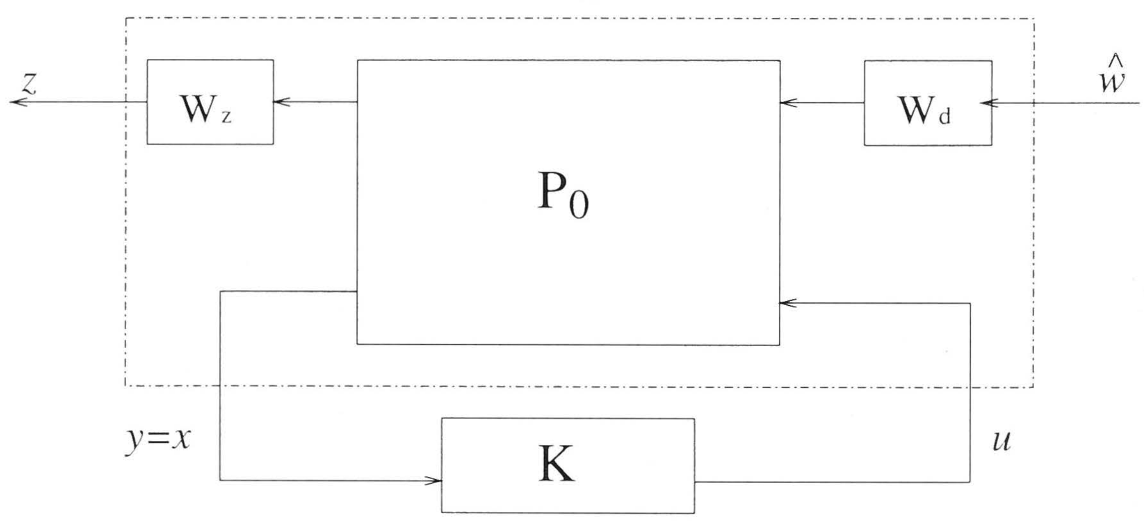

Figure 2.7: The structure of a mixed sensitivity problem

components, $w_{11} = x_{w11}$ and $w_{12} = C_{w2}x_{w2}$.

$$z = e' = x_{w_{11}} + u. \qquad (2.5^*)$$

2.5 Simplification of the state feedback HJ PDE for the nonlinear disturbance suppression problem under comprehensive stability

Here, we extend the concept of so-called comprehensive stability [49] to the nonlinear $\mathcal{H}_\infty$ problem. This includes the nonlinear disturbance rejection problem, which contains un-stabilisable states. The constant disturbance rejection problem (as we have formulated it) is a nonstandard $\mathcal{H}_\infty$ problem, because $x_{w_{11}}$ is not stabilisable from u.

First we introduce the standard nonlinear state feedback $\mathcal{H}_\infty$ control problem: See

to provide another choice of z.

Fig 2.7, let the state space model for plant P be:

$$\begin{cases} \dot{x} &= A(x) + B_1(x)\hat{w} + B_2(x)u \\ z &= C_1(x) + D_{12}(x)u \\ y &= x. \end{cases} \tag{2.6}$$

The standard state feedback $\mathcal{H}_\infty$ control problem is to find a controller $u = K(x)$ which makes the closed loop (P,K) γ-dissipative and internally stable, see [28]. Internal stability is the condition that $x(t) \to 0$ as $t \to 0$ for all $x(0)$ and $\hat{w} \in L_2[0, \infty)$.

Theorem 2.1 Consider the system defined by equation (2.6), and suppose that $\exists \alpha, \beta : \alpha I \geq E_1 = D_{12}^T D_{12} \geq \beta I > 0$ for all x. Suppose one can find a strictly positive proper smooth function V of x, such that $V(x) > 0$ for $x \neq 0$, $V(0) = 0$ and which (a) satisfies the state feedback Hamilton Jacobi PDE (HJ PDE)

$$\begin{aligned} & \nabla_x V \left(A - B_2 E_1^{-1} D_{12}^T C_1\right) \\ + \; & \frac{1}{2} \nabla_x V \left(\gamma^{-2} B_1 B_1^T - B_2 E_1^{-1} B_2^T\right) \nabla_x V^T \\ + \; & \frac{1}{2} C_1^T \left(I - D_{12} E_1^{-1} D_{12}^T\right) C_1 = 0. \end{aligned} \tag{2.7}$$

and (b) makes the vector field

$$A - B_2 E_1^{-1} D_{12}^T C_1 + \left(\gamma^{-2} B_1 B_1^T - B_2 E_1^{-1} B_2^T\right) \nabla_x V^T$$

asymptotically stable. Then the central controller for the state feedback problem, which guarantees γ-dissipativity and internal stability, is defined as:

$K^\star(x) = -E_1(x)^{-1}[D_{12}(x)^T C_1(x) + B_2(x)^T \nabla_x V(x)^T]$.

Furthermore, even if (b) is not fulfilled, the closed-loop satisfies the dissipation inequality for all T, $x(0)$, and $\hat{w}(.)$:

$V(x(t)) + \frac{1}{2} \int_0^T |z(t)|^2 dt \leq \gamma^2 \frac{1}{2} \int_0^T |\hat{w}(t)|^2 dt + V(x(0))$.

Proof See [28]. ∎

For some mixed sensitivity problems, such as (2.5), there exist some un-stabilisable states, so it is obvious that no stabilising solution for the HJ PDE exists. In order to get around this obstacle, we extend the concepts of comprehensive stability and essential stability [50] to nonlinear systems.

Definition 2.2 The closed-loop system (P, K) in Fig 2.7 is essentially stable if the interconnection of the physical plant P_0 and controller K is internally stable, or equivalently, if the only non-internally-stable modes of (P, K) are those associated with the weighting.

The motivation is that the weighting is not present in any physical sense, while P_0 and K are physically present.

Definition 2.3 The closed-loop system (P, K) in Fig 2.7 is said to be comprehensively stable if it is essentially stable, and the closed-loop from $\hat{w}$ to z is γ-dissipative. When this is the case, K is called a comprehensively stabilising controller.

As a first step towards adjusting Theorem 2.1 to cope with un-stabilisable states, we present Theorem 2.7 below. This theorem uses the concept of zero detectability; we now define this concept for the system.

Definition 2.4 The system (of Figure 2.7) with input $\hat{w}$ and output z is said to be zero-detectable if the conditions that $\hat{w}(t) = 0$ and z(t)=0 for all $t \geq 0$, are sufficient to imply that $\lim_{t\to\infty} x(t) = 0$.

We present a lemma as follows, which will be needed for the main stability theorem. It comes from a simple extension of La Salle's invariance principle [62].

Definition 2.5 Consider the system (of Figure 2.7) with $\hat{w} = 0$. Let $x = [x_w^T\ x_s^T]^T$. Define Π as the projection from $\mathcal{R}^{\dim(x)}$ to $\mathcal{R}^{\dim(x_s)}$ in the obvious way by $\Pi(\begin{bmatrix} x_w^T\ x_s^T \end{bmatrix}^T) = x_s$.

Lemma 2.6 Consider the system (of Figure 2.7) with $\hat{w} = 0$. Let $V(x)$ be a scalar function with continuous partial derivatives, and $\mathcal{B}_r$ be the open set defined as $\{x : V(x) < r\}$. Assume that for a fixed but arbitrary $r \in \mathcal{R}$, $\Pi\mathcal{B}_r$ is a bounded set where the projection operator Π is defined in Definition 2.5, and that also within $\mathcal{B}_r$ the following conditions hold

- $\dot{V}(x) \leq 0$
- $V(x) > 0$ for $x_s \neq 0$ and $V(x) = 0$ for $x_s = 0$,

- for every trajectory of x starting from $x(0)$ within $\mathcal{B}_r$, there is a bound for $x(t)$ (which may possibly depend on $x(0)$).

Let $\mathcal{N}$ be the set of all points within $\mathcal{B}_r$ where $\dot{V}(x) \equiv 0$ and let $\mathcal{M}$ be the largest invariant set within $\mathcal{N}$. Then for every possible $x(0)$ in $\mathcal{B}_r$, as $t \to \infty$, $x(t) \to \mathcal{M}$ and consequently every associated projection $x_s = \Pi(x)$ tends to $\mathcal{M}_s = \Pi(\mathcal{M})$.

Proof The proof is in Appendix 2.8. ∎

Theorem 2.7 Consider the system defined by equation (2.6), and suppose that $\exists \alpha, \beta : \alpha I \geq E_1 = D_{12}^T D_{12} \geq \beta I > 0$ for all x. Suppose also that $u = K(x)$ for some K such that $K(0) = 0$. Suppose that the state vector of P is of the form $[x_w^T \; x_s^T]^T$, in which the components x_s are stabilisable from u and the components x_w are associated only with *weights* and are not necessarily stabilisable. Then the closed-loop system (P, K) will be comprehensively stable, given the following conditions are satisfied:

- There exists a storage function V, such that $V(x) > 0$ if $x_s \neq 0$ and $V(x) = 0$ if $x_s = 0$, which satisfies the dissipative inequality:

$$\dot{V}(x) \leq \frac{1}{2}[\gamma^2 ||\hat{w}||^2 - ||z||^2]. \tag{2.8}$$

- The states x_s are *zero*-detectable.

Proof The proof is in Appendix 2.9. ∎

Next, let us go back to the nonlinear disturbance suppression problem. We present a theorem which gives a sufficient condition for system (2.5) to be comprehensively stabilised. This condition is relatively moderate, and easier to check than that in Theorem 2.1.

Theorem 2.8 Consider the system described by equation (2.5). Suppose that the state vector of the plant P is of the form $[x_{w_{11}} \; x_s^T]^T = [x_{w_{11}} \; x_{w12}^T \; x_{w_2}^T \; x_0^T]^T$, where the sub-state x_s is zero-detectable. If there exists a function $\bar{V}(x)$, such that $\bar{V}(x) > 0$ if $x_s \neq 0$ and $\bar{V}(x) = 0$ if $x_s = 0$, which satisfies the following HJ PDE:

$$\nabla_{x_s}\bar{V}\bar{A} + \frac{1}{2}\nabla_{x_s}\bar{V}\left(\gamma^{-2}\bar{B}_1\bar{B}_1^T - \bar{B}_2\bar{E}_1^{-1}\bar{B}_2^T\right)\nabla_{x_s}\bar{V}^T = 0, \tag{2.9}$$

then the system (2.5) can be comprehensively stabilised by the central controller $K^\star(x_s)$, defined as:

$$K^\star(x_s) = -\bar{E}_1(x_s)^{-1}[\bar{D}_{12}(x_s)^T C_1(x) + \bar{B}_2(x_s)^T\nabla_{x_s}\bar{V}(x)^T] \tag{2.10}$$

In the above equations the terms are given by

$$\bar{A} = \bar{A}(x_s) = \begin{bmatrix} A_{w12}x_{w12} \\ A_{w2}x_{w2} \\ A(x_0) + B_1(x_0)C_{w12}x_{w12} \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} B_{w12} & 0 \\ 0 & B_{w2} \\ 0 & 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 \\ 0 \\ B_1(x_0) \end{bmatrix},$$

$C_1(x) = x_{w_{11}}, \bar{D}_{12} = D_{12}, and\ \bar{E}_1 = \bar{D}_{12}^T\bar{D}_{12}$.

Proof The proof is in Appendix 2.10. ■

Remarks:

- By applying Theorem 2.8, we can achieve a $(n + n_{w_{12}} + n_{w_2})$-th order HJ PDE (2.9) instead of a $(n + n_{w_{12}} + n_{w_2} + 1)$-th order HJ PDE (2.7) to construct the comprehensively stabilising controller for the disturbance suppression problem.
- It is obvious that $\bar{V} = 0$ is one solution of the HJ PDE (2.9); since when $\bar{V} = 0$, then $z = 0$, and the γ-dissipativity condition is satisfied. However for such a $\bar{V}$, stability is not necessarily guaranteed. On the other hand, if P_0 is a stable plant, the stability requirement is automatically satisfied with $\bar{V} = 0$. This means that the state feedback controller is just $-x_{w_{11}}$, and therefore output feedback just needs to feed back the estimate of the state $x_{w_{11}}$. This greatly simplifies the design of the output feedback $\mathcal{H}_\infty$ controller for this problem, because we only need to observe the state $x_{w_{11}}$.

Now we give a simple example to illustrate our method.

Example Consider the problem of Fig 2.4. We suppose that $G_{w_1}(s) = G_{w_2}(s) = 1$, and that P_0 is given by

$$P_0 \begin{cases} \dot{x}_2 &= -ax_2^3 + \hat{w}_1 + u \\ y &= x_2 + \hat{w}_2. \end{cases} \tag{2.11}$$

From equation (2.5) the combined system (i.e. P_0 with weights) is

$$P \begin{cases} \dot{x}_1 &= \hat{w}_1 \\ \dot{x}_2 &= -ax_2^3 + x_1 + u \\ z &= x_1 + u \\ y &= x_2 + \hat{w}_2 \end{cases} \tag{2.12}$$

From Theorem 2.8, we get the HJ PDE

$$-ax_2^3\bar{V}_{x_2} - \frac{1}{2}\bar{V}_{x_2}^2 = 0.$$

There are two solutions for $\bar{V}_{x_2}$, namely $\bar{V}_{x_2} = 0$ and $\bar{V}_{x_2} = -2ax_2^3$. From the initial condition $\bar{V}(0) = 0$, we obtain the two solutions:

$$\bar{V}_1(x_2) = 0 \;\text{ and }\; \bar{V}_2(x_2) = -\frac{1}{2}ax_2^4.$$

For the case where $a < 0$, it follows that $\bar{V}_2(x_2) > 0$ and $\bar{V}_2(x_2)$ satisfies all the conditions of Theorem 2.8. Hence, the controller $u = -x_1 + 2ax_2^3$ comprehensively stabilises the system (2.12).

For the case where $a > 0$, it is obvious that $\bar{V}_2(x_2)$ cannot be a storage function since $\bar{V}_2(x_2) < 0$. Although $\bar{V}_1(x_2) = 0$ can be a storage function, it does not satisfy the conditions of Theorems 2.7 and 2.8 since $\bar{V}_1(x_2) = 0$ even when $x_s \neq 0$. Fortunately, we can directly check that by using $\bar{V}_1(x_2) = 0$, we can achieve a comprehensively stabilising controller for the system (2.12). Firstly, we note that $\bar{V}_1(x_2) = 0$ satisfies the γ-dissipativity condition. Secondly, because $a > 0$, it follows that P_0 is stable. From these two facts, we can conclude that the controller $u = -x_1$, which is constructed by using $\bar{V}_1(x_2) = 0$ as a storage function, can comprehensively stabilise the system (2.12).

2.6 The structure of the disturbance suppression output feedback controller for nonlinear plants

In this section, we shall discuss the use of output feedback rather than state feedback to achieve constant disturbance suppression. We first show how one might use a nonlinear observer in conjunction with a state-feedback $\mathcal{H}_\infty$ controller in order to develop an output feedback controller. The form of solution suggests that an output feedback controller which rejects constant disturbances may contain an integrator. We aim to confirm that the controller normally acquires an integrator, a phenomenon well known in the linear case, and the basis of classical constant disturbance suppression ideas.

Nonlinear $\mathcal{H}_\infty$ output feedback control is particularly difficult. The standard solution of the linear $\mathcal{H}_\infty$ output feedback control problem normally depends upon solving two Riccati equations [84]. One of these, which arises in the state feedback control problem, is replaced by an HJ PDE in the nonlinear case. The other, however, is replaced by a still more complicated equation (involving an information state), see [28]. Practical approaches to solution of this latter equation are so far lacking.

As an alternative, one can draw on ideas of nonlinear observer theory [8] [39] [43], and substitute a state estimate $\hat{x}$ instead of the state x in a state feedback controller, retrospectively checking the γ-dissipativity and stability of the closed-loop system. In this case, the controller remains finite-dimensional, which is not normally the case when information state methods are used.

In the linear case, the disturbance rejection output feedback controller necessarily includes an integrator. We now investigate the output feedback controller structure for the nonlinear case. We first define a notion of internal stability.

> **Definition 2.9** A closed loop system is internally stable around all constant operating points if when subjected to inputs composed of the sum of a constant signal plus signal in $\mathcal{L}_2$, all internal states $x(t)$ of the closed loop system become the sums of constant signals plus a signals in $\mathcal{L}_2$. Output stability is defined similarly.

This definition reduces to the standard notion of stability in the linear case. By analogy with the linear case we shall adopt the following definition.

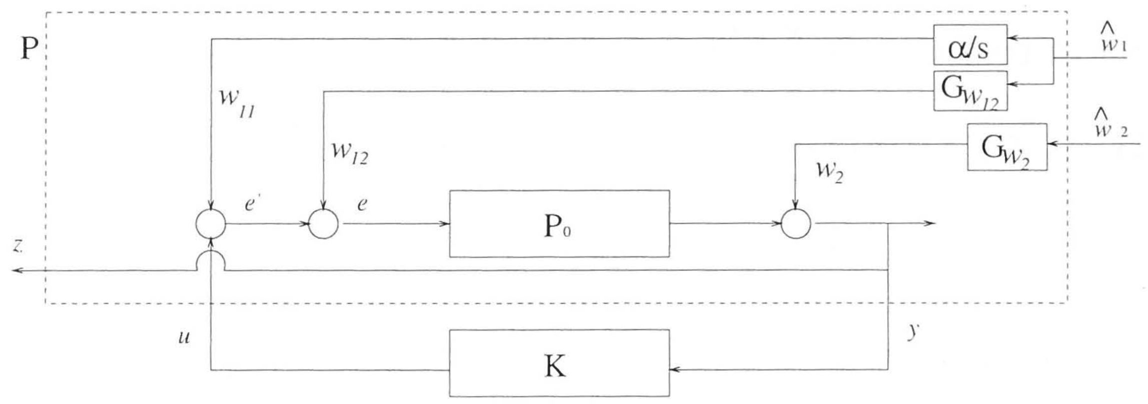

Figure 2.8: Another choice for z (See Theorem 2.12)

Definition 2.10 A nonlinear system contains an integrator iff there exist some initial conditions for the state, and some input signal in $\mathcal{L}_2$ which results in the output being the sum of a non-zero constant signal plus a signal in $\mathcal{L}_2$.

Theorem 2.11 Consider the constant disturbance suppression problem described by equations (2.5) and (2.5*) and depicted in Figure 2.6. Suppose that an output feedback $\mathcal{H}_\infty$ controller exists, such that the resultant closed loop is both internally stable and output stable around all constant operating points in the sense of Definition 2.9. Then assuming that the plant P_0 does not contain an integrator in the sense of Definition 2.10, then the controller must contain an integrator.

Proof The proof is in Appendix 2.11. ∎

Theorem 2.12 Consider the constant disturbance suppression problem described by equations (2.5) and depicted in Figure 2.8. If we choose $z = [y]$, and suppose that an output feedback $\mathcal{H}_\infty$ controller exists, such that the resultant closed loop is both internally stable and output stable around all constant operating points in the sense of Definition 2.9. Then the controller must contain an integrator in the sense of Definition 10, regardless of whether or not the plant contains an integrator.

Proof The proof is in Appendix 2.12. ∎

There is a relevant "nonlinear internal model" principle expressed in [19] and [35], that allows for an exogenous marginally stable system defining the disturbance or tracking signal. The key difference is that, Theorem 2.11 and Theorem 2.12 consider the input constant disturbance rejection under additive norm bounded model uncertainty, while the "nonlinear internal model" principle expressed in [19] and [35] mainly copes with the parametric model uncertainty.

2.7 Conclusion

This chapter presents a modest extension of nonlinear $\mathcal{H}_\infty$ theory in order to solve the constant disturbance rejection problem. We have suggested a nonlinear extension of a concept introduced for the corresponding linear problem, that of the "*comprehensively stabilising*" controller, and have achieved an order reduced HJ PDE for the state feedback problem. Furthermore, we draw the conclusion that the output feedback controller normally must contain an integrator for constant disturbance suppression. This method improves our intuitive understanding of the linear problem.

2.8 Proof of Lemma 2.6

Proof Since $\dot{V}(x) \leq 0$ then $V(x(t)) \leq V(x(0)) = v$ as $t \to \infty$ so that $x(t) \in \mathcal{B}_v$ for $t \geq 0$. Since by hypothesis $V(x) \geq 0$ for all $x \in \mathcal{B}_r$, it follows that $V(x(t))$ has a limit l as $t \to \infty$, where $l \leq v$.

Let Γ be the (positive) limiting set of $x(t)$. Note that Γ is not empty due to the boundedness of the trajectories of $x(t)$ on $\mathcal{B}_r$. By the continuity of $V(x)$ we conclude that $V(x_\Gamma) = l$ for all $x_\Gamma \in \Gamma$ and that therefore $\Gamma \subset \mathcal{B}_l$ and $\dot{V}(x) \equiv 0$ on Γ. Since Γ is an invariant set it follows that $\Gamma \subset \mathcal{M}$. Since $x(t)$ remains bounded within $\mathcal{B}_v$, it follows that $x \to \mathcal{M}$ as $t \to \infty$ and the theorem conclusion follows. ∎

2.9 Proof of Theorem 2.7

Proof From equation (2.8), we can calculate that the closed-loop from $\hat{w}$ to z is γ-dissipative. Now we only need to prove that the state x_s is asymptotically stabilised.

Because the inequality (2.8) is satisfied for all $\hat{w}$, for the case when $\hat{w} = 0$; we have

$$\dot{V}(x) \le -||z||^2.$$

Now we appeal to Lemma 2.6. The set of trajectories for which $\dot{V} \equiv 0$ is the set for which $z(t) \equiv 0$. By the theorem hypothesis, $\hat{w} \equiv 0$ and $z = 0$ imply $\lim_{t\to\infty} x_s(t) = 0$.

∎

2.10 Proof of Theorem 2.8

Proof Let $u = \tilde{u} + \hat{u}$, and

$$\hat{u} = -E_1(x)^{-1}D_{12}(x)^T C_1(x) = -\bar{E}_1(x_s)^{-1}\bar{D}_{12}(x_s)^T C_1(x) = x_{w_{11}} \tag{2.13}$$

Then, equation (2.5) and (2.5 *) together become:

$$\begin{cases} \dot{x}_{w_{11}} = \alpha \hat{w}_1 & (x_{w_{11}} \in R) \\ \dot{x}_{w_{12}} = A_{w_{12}} x_{w_{12}} + B_{w_{12}} \hat{w}_1 & (x_{w_{12}} \in R^{n_{w_{12}}}) \\ \dot{x}_{w_2} = A_{w_2} x_{w_2} + B_{w_2} \hat{w}_2 & (x_{w_2} \in R^{n_{w_2}}) \\ \dot{x}_0 = A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}} + B_1(x_0)\tilde{u} & \\ z = \tilde{u} & \\ y = C_2(x_0) + C_{w_2} x_{w_2} + D_{w_2} \hat{w}_2 & \end{cases} \tag{2.14}$$

For the system defined by equation (2.14), set

$$\hat{A}(x) = \begin{bmatrix} 0 \\ A_{w_{12}} x_{w_{12}} \\ A_{w_2} x_{w_2} \\ A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}} \end{bmatrix},$$

$$\hat{B}_1(x) = \begin{bmatrix} \alpha & 0 \\ B_{w_{12}} & 0 \\ 0 & B_{w_2} \\ 0 & 0 \end{bmatrix},$$

$$\hat{B}_2(x) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_1(x_0) \end{bmatrix},$$

$$\hat{C}_1(x) = 0,$$

$$\hat{D}_{12}(x) = 1,$$

$$and \;\; \hat{E}_1 \triangleq \hat{D}_{12}^T \hat{D}_{12}.$$

Then according to Theorem 2.1, the HJ PDE for the above system is:

$$\begin{aligned}
&\left[V_{x_{w_{11}}}\ V_{x_{w_{12}}}\ V_{x_{w_2}}\ V_{x_0}\right]\left(\hat{A}(x) - 0\right) + \frac{1}{2}\left[V_{x_{w_{11}}}\ V_{x_{w_{12}}}\ V_{x_{w_2}}\ V_{x_0}\right] \\
&\left(\gamma^{-2}\begin{bmatrix} \alpha & 0 \\ B_{w_{12}} & 0 \\ 0 & B_{w_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \alpha & B_{w_{12}}^T & 0 & 0 \\ 0 & 0 & B_{w_2}^T & 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_1(x_0) \end{bmatrix}\left[0\ 0\ 0\ B_1(x_0)^T\right]\right) \\
&\begin{bmatrix} V_{x_{w_{11}}} \\ V_{x_{w_{12}}}^T \\ V_{x_{w_2}}^T \\ V_{x_0}^T \end{bmatrix} = 0.
\end{aligned} \tag{2.15}$$

This may also be expressed as

$$\begin{aligned}
& V_{x_{w_{12}}} A_{w_{12}} x_{w_{12}} + V_{x_{w_2}} A_{w_2} x_{w_2} + V_{x_0}\left[A(x_0) + B_1(x_0) C_{w_{12}} x_{w_2}\right] \\
+ \;& \frac{1}{2}\gamma^{-2}\left[V_{x_{w_{11}}}\ V_{x_{w_{12}}}\ V_{x_{w_2}}\ V_{x_0}\right]\begin{bmatrix} \alpha^2 & \alpha B_{w_{12}} & 0 & 0 \\ \alpha B_{w_{12}} & B_{w_{12}} B_{w_{12}}^T & 0 & 0 \\ 0 & 0 & B_{w_2} B_{w_2}^T & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} V_{x_{w_{11}}} \\ V_{x_{w_{12}}}^T \\ V_{x_{w_2}}^T \\ V_{x_0}^T \end{bmatrix} \\
- \;& \frac{1}{2}\left[V_{x_{w_{11}}}\ V_{x_{w_{12}}}\ V_{x_{w_2}}\ V_{x_0}\right]\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1(x_0) B_1^T(x_0) \end{bmatrix}\begin{bmatrix} V_{x_{w_{11}}} \\ V_{x_{w_{12}}}^T \\ V_{x_{w_2}}^T \\ V_{x_0}^T \end{bmatrix} \\
= \;& 0.
\end{aligned} \tag{2.16}$$

Now with V the solution of equation (2.9), we can verify that $V(x_{w_{11}}, x_{w_{12}}, x_{w_2}, x_0) = \bar{V}(x_{w_{12}}, x_{w_2}, x_0)$ satisfies (2.16). For with this identification, $\bar{V}_{x_{w_{11}}} = 0$, and equation (2.16) becomes:

$$
\begin{aligned}
& \begin{bmatrix} \bar{V}^T_{x_{w_{12}}} \\ \bar{V}^T_{x_{w_2}} \\ \bar{V}_{x_0} \end{bmatrix} \begin{bmatrix} A_{w_{12}} x_{w_{12}} \\ A_{w_2} x_{w_2} \\ A(x_0) + B_1(x_0) C_{w_{12}} x_{w_{12}} \end{bmatrix} \\
+ & \frac{\gamma^{-2}}{2} \begin{bmatrix} \bar{V}_{x_{w_{12}}} & \bar{V}_{x_{w_2}} & \bar{V}_{x_0} \end{bmatrix} \begin{bmatrix} B_{w_{12}} B^T_{w_{12}} & 0 & 0 \\ 0 & B_{w_2} B^T_{w_2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{V}^T_{x_{w_{12}}} \\ \bar{V}^T_{x_{w_2}} \\ \bar{V}^T_{x_0} \end{bmatrix} \\
- & \frac{1}{2} \begin{bmatrix} \bar{V}_{x_{w_{12}}} & \bar{V}_{x_{w_2}} & \bar{V}_{x_0} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_1(x_0) B_1^T(x_0) \end{bmatrix} \begin{bmatrix} \bar{V}^T_{x_{w_{12}}} \\ \bar{V}^T_{x_{w_2}} \\ \bar{V}^T_{x_0} \end{bmatrix} \\
= & \ 0.
\end{aligned}
\tag{2.17}
$$

This is identical to equation (2.9), which is true by hypothesis. Since the solution $\bar{V}(x)$ has the property that $\bar{V}(x) > 0$ if $x_s \neq 0$, $\bar{V}(x) = 0$ if $x_s = 0$, then from Theorem 2.1 we conclude that equation (2.15) has a solution that makes the closed-loop $(P, K^\star)$ γ-dissipative without necessarily satisfying the closed-loop asymptotic stability condition of Theorem 2.1.

Because x_s is zero-detectable by hypothesis, then from the γ-dissipativity property that we have demonstrated immediately previously, and from application of Theorem 2.7, we conclude that the closed loop (P, K) is comprehensively stabilised. ∎

2.11 Proof of Theorem 2.11

Proof Consider the case where $\hat{w}_1, \hat{w}_2 \in \mathcal{L}_2$ and the input w_{11} is composed of the sum of a non-zero constant signal plus a signal in $\mathcal{L}_2$, and $w_2 \in \mathcal{L}_2$. Since there is an $\mathcal{H}_\infty$ controller, the signal z must obey the dissipation inequality with respect to $\hat{w}_1$ and $\hat{w}_2$, and hence $z \in \mathcal{L}_2$. By the properties of w_{11} and z (See Fig 2.6), it follows that the demanded controller output u can be described as a *non-zero* constant signal plus a signal in $\mathcal{L}_2$

Since the closed loop system is stable, it follows that both the plant states x_p and the observed outputs must be able to be described as constant signals plus signals in $\mathcal{L}_2$. Because of our assumption that the nonlinear plant does not contain an integrator, the observed output y must be a *zero* constant signal plus a signal in $\mathcal{L}_2$.

We now observe that the controller K has an input $y \in \mathcal{L}_2$ and an output u which is a *non-zero* constant signal plus a signal in $\mathcal{L}_2$. By Definition 2.10, the controller contains an integrator.

∎

2.12 Proof of Theorem 2.12

Proof Consider the case where $\hat{w}_1, \hat{w}_2 \in \mathcal{L}_2$ and the input w_{11} is composed of the sum of a non-zero constant signal plus a signal in $\mathcal{L}_2$, and $w_2 \in \mathcal{L}_2$. Since there is an $\mathcal{H}_\infty$ controller, the signal z must obey the dissipation inequality with respect to $\hat{w}_1$ and $\hat{w}_2$, and hence $z \in \mathcal{L}_2$.

Because the closed loop system is stable, it follows that both the plant states x_p and the observed outputs must be able to be described as constant signals plus signals in $\mathcal{L}_2$. Because of the observed output y is chosen as z, y must be a *zero* constant signal plus a signal in $\mathcal{L}_2$.

By the properties of w_{11} and z (See Fig 2.8), it follows that the demanded controller output u can be described as a *non-zero* constant signal plus a signal in $\mathcal{L}_2$ [2].

We now observe that the controller K has an input $y \in \mathcal{L}_2$ and an output u which is a *non-zero* constant signal plus a signal in $\mathcal{L}_2$. By Definition 2.10, the controller contains an integrator.

∎

[2] If u is *zero* constant signal plus a signal in $\mathcal{L}_2$, then the input of the plant P_0 will be the *non-zero* constant signal of w_{11} plus a signal in $\mathcal{L}_2$. However, the output y of the plant P_0 is a *zero* constant signal plus a signal in $\mathcal{L}_2$. That means the plant has already been possessed of the ability of constant disturbance rejection. Certainly, we need not to design a controller to deal with the constant disturbance rejection problem for such plant.

Chapter 3

Disturbance suppression for nonlinear systems design using singular perturbation theory

A relatively practical method of suppressing the effect of constant disturbances on nonlinear systems is presented in this chapter. By adding an integrator to a stabilising controller, it is possible to achieve both constant disturbance rejection and zero tracking error. Sufficient conditions for the rejection of a constant input disturbance are given. We give both local and global conditions such that the inclusion of an integrator in the closed loop maintains closed loop stability. The analysis is based on singular perturbation theory. Furthermore, we also present some alternative locations for adding an integrator into the closed loop system and extend these methods to deal with Multiple-input Multiple-output nonlinear systems. Finally, we implement our method in the control of a simulated helicopter model. The simulation results show that this method achieves satisfactory performance.

3.1 Introduction

An important objective of control system design is to minimise the effects of external disturbances. The problem of disturbance rejection (especially constant disturbance rejection) arises in many industrial fields, such as motion-control, active noise control and vibration control. For linear systems, the classical method of rejecting a constant disturbance is to include an integrator in the controller. This chapter extends this idea to nonlinear systems, using singular perturbation methods to guarantee stability.

Although the method presented in this chapter extends classical methods for linear constant disturbance rejection, it is also related to nonlinear $\mathcal{H}_\infty$ methods presented in the last chapter[72]. The last chapter extended the concept of comprehensive stability for linear systems [50] [49] to deal with the nonlinear constant disturbance suppression problem. As an $\mathcal{H}_\infty$ mixed sensitivity problem, the constant disturbance suppression problem is, a nonstandard due to the existence of un-stabilisable states.

The main bottleneck for nonlinear state feedback $\mathcal{H}_\infty$ control, which is similar to the problem encountered in nonlinear optimal control, is the need to solve a Hamilton-Jacobi (HJ) partial differential equation (PDE)[45] [9]. In last chapter, we [72] [75] presented a method of simplifying (via order reduction) the Hamilton-Jacobi partial differential equation for the nonlinear disturbance rejection problem by using the concept of *comprehensive stability*, a concept which is extended from the linear case [49]. Because the states which are related to the disturbance are not directly measurable, they cannot be directly used. This forces us to consider nonlinear $\mathcal{H}_\infty$ output feedback control.

Nonlinear $\mathcal{H}_\infty$ output feedback control is particularly difficult. The standard solution of the linear $\mathcal{H}_\infty$ output feedback control problem normally involves solving two Riccati equations [84]. One of these, which arises in the state feedback control problem, is replaced by a Hamilton-Jacobi partial differential equation in the nonlinear case. The other, however, is replaced by a still more complicated equation (involving an information state)[28]. Practical approaches to the solution of this latter equation are so far lacking. Alternatively, one can draw on ideas of nonlinear observer theory [39] [43], and replace the state x with a state estimate $\hat{x}$ in a state feedback controller, retrospectively checking the γ-dissipativity and stability of the closed-loop system. In this case, the controller remains finite-dimensional, which is not always the case when information state methods are used.

In last chapter, we also demonstrate that for disturbance suppression, an output feedback controller must contain an integrator [72] [3]. In this chapter we ask whether we can directly add an integrator to an already existing controller to achieve constant disturbance rejection, while still retaining the stability of the system. Often that

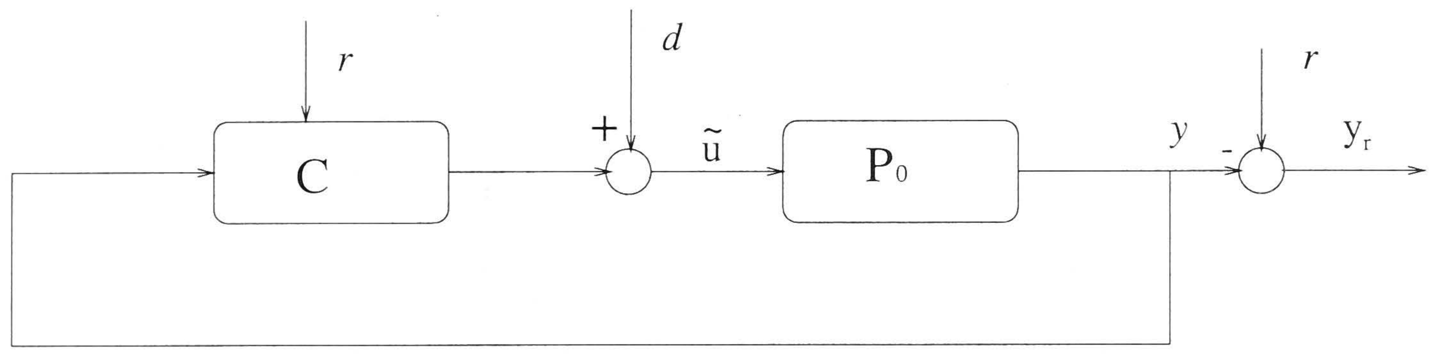

Figure 3.1: A nonlinear constant disturbance suppression problem

would be both a simpler and more practical way to deal with the nonlinear constant disturbance suppression problem [3].

This chapter not only gives the affirmative answer but also suggests several locations where an integrator with low gain away from DC ($\frac{\epsilon}{s}$ for short) maybe included, in order to deal with the constant input disturbance rejection problem. Furthermore, this method can also be applied to cope with the constant reference tracking problem, even for nonlinear MIMO systems, such as the helicopter system of [41].

In the next section, we give a description of the problem. In Section 3.3, we present a proof that an exponentially stabilising nonlinear controller appropriately augmented with a small integrator (a linear transfer function $\frac{\epsilon}{s}$) can yield constant disturbance suppression. In Section 3.4, we will give both local and global conditions for the existence of a gain of the integrator that is sufficiently small to guarantee stability. Section 3.5, by using singular perturbation methods, gives a upper bound on a value of the gain that guarantees closed loop stability. In Section 3.6, we suggest alternative locations for adding an integrator into the system. Section 3.7 extends our method to deal with the constant disturbance rejection problem and constant reference tracking problem for Multiple-input Multiple-output(MIMO) systems. Finally in Section 3.8, we present simulation results obtained by implementing constant disturbance rejection and zero steady state tracking error control for a helicopter model by using this method.

3.2 Problem description

Firstly, let us consider a nonlinear input disturbance rejection problem as shown in Figure 3.1. This depicts a nonlinear single-input single output (SISO) system (We will extend our methods to MIMO systems later). It consists of the interconnection of

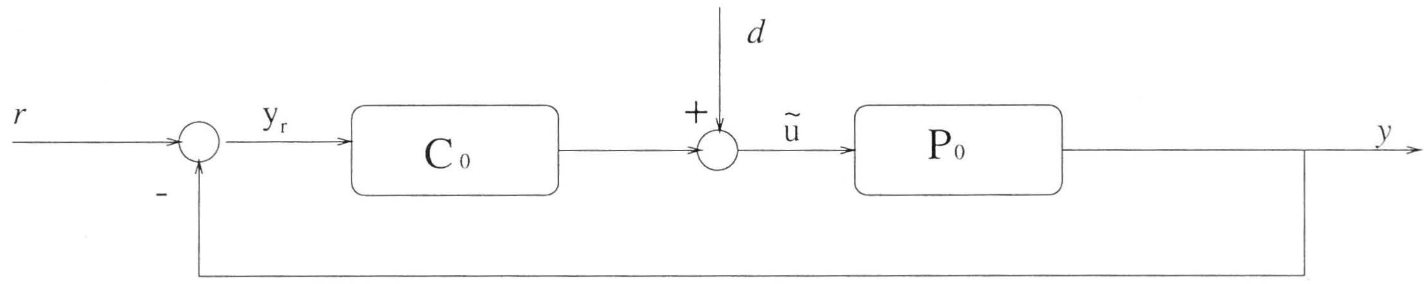

Figure 3.2: A nonlinear system with an existing stabilising controller C_0

a nonlinear plant P_0 and controller C, forced by a constant command signal r, as well as a constant input disturbance d. Here, y_r is the reference tracking error, and $\tilde{u}$ is the input to the plant. What we are concerned with here is how to design a controller C which possesses the ability to both reject a constant input disturbance d, and to give zero steady state tracking error for a constant reference input r.

More precisely, we consider the question of how we might modify a pre-existing controller C_0 not achieving these properties, so that the properties are secured throughout the modification (See Figure 3.2).

In the case of a linear plant, the classical method employed to reject a constant disturbance is to include an integrator in the controller. Here, we extend this idea to deal with the nonlinear constant disturbance rejection problem.

Consider Figure 3.3. Suppose that we have already designed a controller C_0 which stabilises the plant P_0 (Later, we shall be precise concerning the type of stability). We then augment the closed loop with the addition of a small gain integrator. The original controller C_0 and small gain integrator $\frac{\epsilon}{s}$ in Case 1 of Figure 3.3 represents a solution to the problem of designing C in Figure 3.1. Then, the interconnection is equivalent to a single stable plant P as shown in Figure 3.3. By stating that the two cases in Figure 3.3 are equivalent, we mean that if the exogenous input signals d and r in the two cases are equal, then all labelled signals (including the output signals) will also be equal(given suitable matching of initial conditions, or after decay of initial condition effects). Hence, we can focus our attention on the simplified second case.

In the second case of Figure 3.3, we suppose that the state equation of the plant P is modelled as follows.

$$P : \left\{ \begin{array}{rcl} \dot{x} & = & f(x, \tilde{u}) \\ y & = & g(x, \tilde{u}). \end{array} \right. \tag{3.1}$$

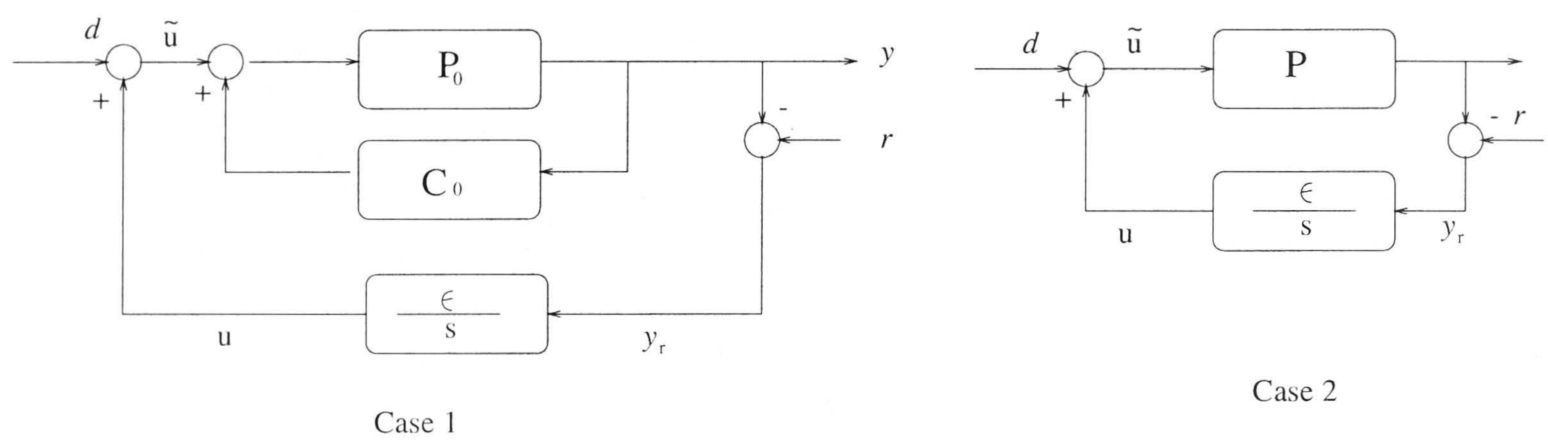

Figure 3.3: Two equivalent cases

If there is no particular declaration in this chapter, we suppose that $f : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^n$ and $g : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^l$ are unbiased in the sense that

$$\left\{ \begin{array}{rcl} f(0,0) & = & 0 \\ g(0,0) & = & 0. \end{array} \right. \tag{3.2}$$

The state equation for the small integrator is expressed as a transfer function block $\frac{\epsilon}{s}$:

$$\frac{\epsilon}{s} : \left\{ \begin{array}{rcl} \dot{\xi} & = & \epsilon y_r \\ u & = & \xi. \end{array} \right. \tag{3.3}$$

In the above, the reference tracking error y_r is equal to $y - r$. We suppose that the disturbance d and the reference input r are both constant.

The following parts of this chapter will focus on two key questions. The first question is whether a controller that is augmented with an integrator will reject the constant disturbance. The second question is how to ensure the stability of the closed loop. Another but nevertheless important question is whether constant reference trajectory following occurs, with zero steady state error.

3.3 Sufficient conditions for constant disturbance rejection

In [72], it was shown that for input disturbance suppression an output feedback $\mathcal{H}_\infty$ controller must contain an integrator in the controller. In this section, we will still start our discussion from the point of view of an $\mathcal{H}_\infty$ treatment.

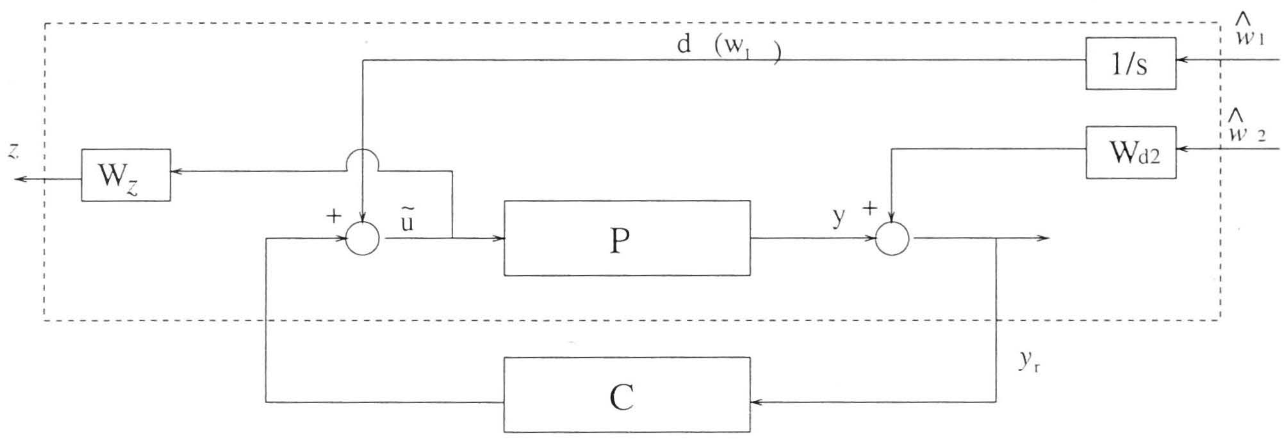

Figure 3.4: The mixed sensitivity $\mathcal{H}_\infty$ form

As in [72], we also extend the constant input disturbance rejection problem to a mixed sensitivity $\mathcal{H}_\infty$ problem (Figure 3.4). We introduce an integrator into one of the input weights(the disturbance weight), and choose cost variable $z = \tilde{u}$. The input $\hat{w}_1$ gives rise to the input disturbance d. The introduction of the input $\hat{w}_2$ can be interpreted as a way of capturing modelling uncertainty or as a reference input signal. Without an integrator weight function, the introduction of $\hat{w}_2$ is necessary for ensuring that the $\mathcal{H}_\infty$ problem is standard. Here, the input weighting function W_{d2} of $\hat{w}_2$ and the output weighting function W_z are both stable.

In order to set up the relationships between input-output stability [79] and Lyapunov stability for this constant disturbance rejection problem, we present a theorem from [79].

We will later identify the controller C in Figure 3.4 with the small gain integrator $(\frac{\epsilon}{s})$.

Definition 3.1 A system is globally exponentially stable (GES) iff there exists a Lyapunov function $U(x) \leq 0$ such that

$$\rho_1|x|^2 \leq U(x) \leq \rho_2|x|^2$$

and with zero input

$$\frac{d}{dt}U(x(t)) \leq -\rho_3|x|^2.$$

Where $\rho_i > 0$, $i = 1, 2, 3$ are suitable scalar constants. If these conditions hold, it follows that there exists some constant $\rho \geq 0$ such that with $x(0) = x_0$,

$|x(t)| \leq \rho|x_0|e^{-\rho_3 t/2}$ for all $t \geq 0$.

By local exponential stability (LES) we mean that this definition is valid at least for x in a neighbourhood of $x = 0$.

Definition 3.2 Consider the nonlinear system of the form

$$\begin{cases} \dot{x} &= f(x,u) \\ y &= g(x,u). \end{cases} \tag{3.4}$$

The system (3.4) is said to be "$\mathcal{L}_p$-stable with finite gain" if there exist constants b_p and $\gamma_p < \infty$ such that $u \in \mathcal{L}_p^m \implies y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p \|u\|_p + b_p$. If $p = 2$, γ_p is said to be the $\mathcal{L}_2$ bound from u to y.

The system (3.4) is said to be "$\mathcal{L}_p$-stable without bias" if there exists a constant $\gamma_p < \infty$ such that $x(0) = 0$, $u \in \mathcal{L}_p^m \implies y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p \|u\|_p$.

The system (3.4) is "small signal $\mathcal{L}_p$-stable without bias" if there exist constants $r_p > 0$ and $\gamma_p < \infty$ such that $x(0) = 0$, $u \in \mathcal{L}_p^m$ with $\|u\|_p \leq r_p \implies y \in \mathcal{L}_p^l$ and $\|y\|_p \leq \gamma_p \|u\|_p$.

As in the linear case, it is possible to establish a connection between these two types of stability [79].

Theorem 3.3 Consider the system described by equation (3.4). Suppose that $f : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^n$ and $g : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^l$ are unbiased in the sense that

$$\begin{cases} f(0,0) &= 0 \\ g(0,0) &= 0. \end{cases} \tag{3.5}$$

which ensures that $x = 0$ is an equilibrium of the **unforced** system

$$\dot{x} = f(x,0). \tag{3.6}$$

Suppose that $x = 0$ is an exponentially stable equilibrium of (3.6), and that f is C^1. Suppose also that f and g are locally Lipschitz continuous at $(0,0)$, that is, suppose there exist finite constants k_f, k_g, r such that

$$\|f(x,u) - f(z,v)\|_2 \leq k_f[\|x - z\|_2 + \|u - v\|_2], \forall (x,u)(z,v) \in B_r, \tag{3.7}$$

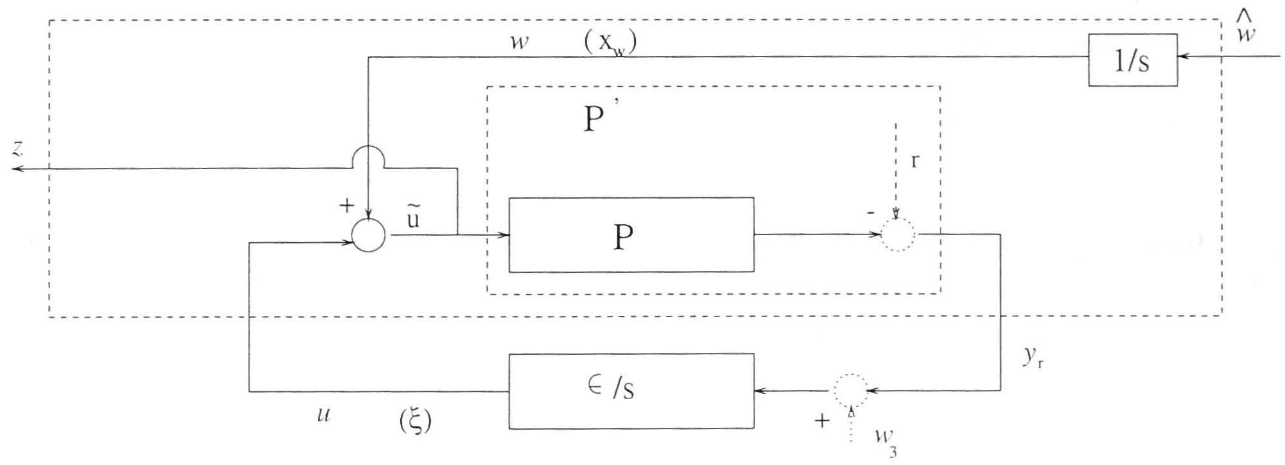

Figure 3.5: The simplified mixed sensitivity $\mathcal{H}_\infty$ form

$$\|g(x,u) - g(z,v)\|_2 \le k_g[\|x-z\|_2 + \|u-v\|_2], \forall (x,u)(z,v) \in B_r. \tag{3.8}$$

Here, B_r is the open ball of the radius r, that is, $B_r = \{x : \|x - x_0\| < r\}$. Then the system (3.4) is small signal $\mathcal{L}_p$-stable without bias for each $p \in [1, \infty)$. If $x = 0$ is a globally exponentially stable equilibrium, and (3.7) and (3.8) hold with B_r replaced by $\mathcal{R}^{(m+n)}$, then the system (3.4) is $\mathcal{L}_p$-stable without bias for each $p \in [1, \infty)$. Furthermore, there exists a Lyapunov function $U(x) \ge 0$ which satisfies the requirements of exponential stability of Definition 3.1, and the gain γ_p is related to the constants ρ_i defining the properties of $U(x)$ by

$$\|y\|_p \le k_g[(\rho_3 k_f / 4{\rho_1}^2{\rho_2}^2) + 1]\|u\|_p.$$

Proof See pages 286-289 of [79]. ∎

Now, let us consider the mixed sensitivity $\mathcal{H}_\infty$ problem depicted by Figure 3.4. One design goal of $\mathcal{H}_\infty$ methods is to ensure that a finite $\mathcal{L}_2$ gain $\gamma_{\hat{w}z}$ exists from input $[\hat{w}_1\ \hat{w}_2]^T$ to output z, in other words to ensure that the system is "$\mathcal{L}_2$-stable with finite gain" (see Definition 3.2.). In this section, in order to emphasise the problem of constant input disturbance rejection as opposed to reference tracking and to simplify our discussion, we will not consider the input $\hat{w}_2$, that is, we set $\hat{w}_2 = 0$. We also assume that the weight function W_z is unity. Because the weighting functions W_{d2} and W_z are both stable, we can use Theorem 3.3 to see that these simplifications will not influence the existence of $\gamma_{\hat{w}z}$ and our further discussion.

We set the controller C in Figure 3.4 to be $\frac{\epsilon}{s}$. The system is then as depicted in Figure 3.5.

Theorem 3.4 Consider the system depicted in Figure 3.5. The plant P and $\frac{\epsilon}{s}$ blocks are respectively described by equations (3.1),(3.2) and (3.3). Suppose that $(0,0)$ is an exponentially stable equilibrium of the **unforced closed loop** $(P, \frac{\epsilon}{s})$. Further, assume that f is C^1, and that f, g are locally Lipschitz continuous at $(0,0)$ with Lipschitz constants k_f and k_g to the Euclidean norm $\|.\|_2$(See Definition 3.1 and Theorem 3.3.).

Then the system depicted in Figure 3.5 is small signal $\mathcal{L}_2$ stable without bias from $\hat{w}$ to z.

If $(0,0)$ is a globally exponentially stable equilibrium, and f, g are globally Lipschitz continuous at $(0,0)$, then the system is $\mathcal{L}_2$ stable without bias.

Proof

Consider Fig 3.5, and suppose that there is an additional input w_3 to the integrator $\frac{\epsilon}{s}$. If we set $w_3 = \frac{\hat{w}}{\epsilon}$ and then replace $\hat{w}$ by zero, the input w_3 is equivalent to the input of the signal $\hat{w}$. That is, we can replace the disturbance input $\hat{w} \in \mathcal{L}_2$ of the system depicted in Fig 3.5 by the equivalent signal $w_3 \in \mathcal{L}_2$. Because $(0,0)$ is an exponentially stable equilibrium of the **unforced** closed loop $(P, \frac{\epsilon}{s})$, then we will see that according to Theorem 3.3 a finite gain $\gamma_{w_3 y}$ from w_3 to z exists.

More precisely, the augmented system with input w_3 and output z can be described as below.

$$\left\{ \begin{array}{rcl} \dot{x} & = & f(x,\xi) \\ \dot{\xi} & = & \epsilon(g(x,\xi) + w_3) \\ z & = & \xi. \end{array} \right. \tag{3.9}$$

Let $x_a = [x^T \ \xi]^T$, then the above equation can be rewritten in the form:

$$\left\{ \begin{array}{rcl} \dot{x}_a & = & f_a(x_a, w_3) \\ z & = & g_a(x_a) \end{array} \right. \tag{3.10}$$

Here, $f_a(x_a, w_3) = \begin{bmatrix} f(x_a) \\ \epsilon(g(x_a) + w_3) \end{bmatrix}$, $g_a(x_a) = \xi$.

Then, $\forall (x_a, w_3), (x_a', w_3') \in \mathcal{R}^{(n+2)}$

$$
\begin{aligned}
& \|f_a(x_a, w_3) - f_a(x_a', w_3')\|_2 \\
= & \left\| \begin{matrix} f(x_a) - f(x_a') \\ \epsilon(g(x_a) - g(x_a')) + \epsilon(w_3 - w_3') \end{matrix} \right\|_2 \\
\leq & \; k_f(\|x - x'\|_2 + \|\xi - \xi'\|_2) + k_g(\|x - x'\|_2 + \|\xi - \xi'\|_2) + \epsilon\|w_3 - w_3'\|_2 \\
\leq & \; \sqrt{2}(k_f + k_g)\|x_a - x_a'\|_2 + \epsilon\|w_3 - w_3'\|_2 \\
\leq & \; k_{f_a}(\|x_a - x_a'\|_2 + \|w_3 - w_3'\|_2).
\end{aligned} \tag{3.11}
$$

Here, $k_{f_a} = max\{\sqrt{2}(k_f + k_g), \epsilon\}$.

Similarly, it is obvious that $\|g_a(x_a) - g_a(x_a')\|_2 \leq k_{g_a}\|x_a - x_a'\|_2$, where $k_{g_a} = 1$.

In view of the assumption that $(0, 0)$ is an exponentially stable equilibrium of the **unforced** closed loop $(P, \frac{\epsilon}{s})$, there exists a Lyapunov function $U(x) \geq 0$, which satisfies the requirements of Definition 3.1. According to Theorem 3.3 the finite gain $\gamma_{\hat{w}_3 z}$ from w_3 to z is $\gamma_{w_3 z} = [(\rho_3 k_{f_a}/4{\rho_1}^2{\rho_2}^2) + 1]$, where the constants ρ_i are defined by the properties of $U(x)$.

Then, in view of the equivalence of the $\hat{w}$ and w_3 described at the beginning of the proof, we see that the bound from $\hat{w}$ to z is $\gamma_{\hat{w}z} \leq \frac{1}{\epsilon}[(\rho_3 k_{f_a}/4{\rho_1}^2{\rho_2}^2) + 1]$.

∎

Remarks:

- The significance of Theorem 3.4 is that it shows that if a controller is augmented with an integrator, and the closed loop is exponentially stable (we will present sufficient conditions for the stability of the closed loop in next section), then input-output stability from $\hat{w}$ to z is ensured. In fact, the $\mathcal{H}_\infty$ norm from $\hat{w}$ to z is less than the given bound $\gamma_{\hat{w}z}$. Note that there is an integrator weight function between $\hat{w}$ and w which ensures that even for a constant disturbance

w, the output signal z is in $\mathcal{L}_2$ and hence asymptotically goes to zero. That is, the controller augmented with a low gain integrator $\frac{\epsilon}{s}$ will reject a constant input disturbance.

- For the mixed sensitivity $\mathcal{H}_\infty$ problem (which includes an additional input $\hat{w}_2$) depicted by Figure 3.4, if a controller contains $\frac{\epsilon}{s}$ and the closed loop is exponentially stable, then it is easy to see that input-output stability from $[\hat{w}_1 \ \hat{w}_2]$ to z is also ensured, based on Theorem 3.3. That is, the controller with $\frac{\epsilon}{s}$ will robustly reject a constant input disturbance.

Note:

Any equilibrium x_e under investigation can be translated to the origin by redefining the state x as $x - x_e$ [60]. For simplicity, in most of the exposition following we will assume that such a translation has already been performed. Thus, for most parts of this chapter, the equilibrium under investigation will be $x_e = 0$. When we need to emphasise a non-zero equilibrium, we will use $x = x_e$ as the equilibrium point instead of $x = 0$.

Consider the plant P' in Figure 3.5. If we have a nonzero constant reference input r, we can consider the original plant P and reference input r to be equivalent to a new plant P' with an equilibrium point (x_e, ξ_e), where $g(x_e, \xi_e) = r$. Sufficient conditions for stability in this situation are that the conditions of Theorem 3.4 are satisfied for the new equilibrium point. We will investigate the constant reference tracking problem in more detail later.

3.4 Guaranteeing stability with integrator augmentation

We have established that a controller augmented with an integrator will reject a constant input disturbance provided that the stability of the overall closed loop is ensured after the augmentation. We are now concerned with the problem of how to design such a controller so as to ensure the stability of the closed loop $(P, \frac{\epsilon}{s})$. In this section, using singular perturbation theory, we will investigate both local and global conditions for the existence of a small scalar ϵ^* such that when $0 < \epsilon < \epsilon^*$ the closed loop (P,$\frac{\epsilon}{s}$) is stable.

Consider the set up of Figure 3.3 described by equations (3.1) and (3.3). If we set the constant input signal r and d to zero in order to analyse the Lyapunov stability of the unforced closed loop $(P, \frac{\epsilon}{s})$, then the state equation for the closed loop $(P, \frac{\epsilon}{s})$ can be expressed as:

$$(P, \frac{\epsilon}{s}) : \begin{cases} \dot{x} &= f(x, \xi) \\ \dot{\xi} &= \epsilon g(x, \xi). \end{cases} \tag{3.12}$$

In order to use the singular perturbation method, we first transform equation (3.12) to its standard singular perturbation form [58].

Let $\tau = \epsilon(t - t_0)$, so that $\tau = 0$ at $t = t_0$. That leads to $\frac{d\tau}{dt} = \epsilon$. Then, we have

$$\begin{cases} \epsilon \frac{d}{d\tau} x &= f(x, \xi) \\ \frac{d}{d\tau} \xi &= g(x, \xi). \end{cases} \tag{3.13}$$

It should be noticed that x is a vector; on the other hand, with a SISO problem, ξ is a scalar.

In order to be consistent with standard singular perturbation notation, we will for the moment use the notation $\dot{x}$ to denote the derivative on the *slow* time scale τ when we analyse singular perturbation models.

Theorem 3.5 (Global conditions for the existence of ϵ^*)

Consider the second case depicted in Figure 3.3 described by equation (3.13) which satisfies the requirement of equation (3.2), and suppose that the following assumptions are satisfied:

(i) The equation $0 = f(x, \xi)$ obtained by setting $\epsilon = 0$ in equation (3.13) implicitly defines a unique C^2 function $x = h(\xi)$.

(ii) For fixed $\xi \in R$, the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x, \xi)$ is Globally Asymptotically Stable (GAS) [60] and Locally Exponentially Stable (LES).

(iii) The equilibrium $\xi = 0$ of the reduced model (slow time scale) $\dot{\xi} = g(h(\xi), \xi)$ is GAS and LES (See Definition 3.1). A sufficient condition is that $g(h(\xi), \xi)\xi < 0$ (when $\xi \neq 0$) and $g(h(\xi), \xi)\xi \leq -\rho|\xi|^2$ for ξ in a neighbourhood of $\xi = 0$.

Then there exists $\epsilon^* > 0$, such that for all $0 \leq \epsilon \leq \epsilon^*$, the equilibrium $(x, \xi) = (0, 0)$ is GAS. Furthermore if the conditions in (ii) and (iii) involve

GES instead of GAS, then the equilibrium $(x, \xi) = (0, 0)$ is GES.

Proof

This follows from Theorem 3.18 in page 90 of [60] and Corollary 2.2 in page 297 of [58].

Consider $V(\xi) = \frac{1}{2}\xi^2$ as a Lyapunov function candidate for the "slow time scale ". Then, $\dot{V}(\xi) = \xi\dot{\xi} = g(h(\xi), \xi)\xi$. This will satisfy the requirements for GAS and LES given that $g(h(\xi), \xi)\xi < 0$ (when $\xi \neq 0$) and $g(h(\xi), \xi)\xi \leq -\rho|\xi|^2$ (for some scalar $\rho > 0$) for ξ in a neighbourhood of $\xi = 0$. On the other hand, the "fast time scale" mode is GAS or ES by assumption.

∎

Remarks:

- Condition (i) will usually be satisfied in practical situations.
- For linear systems the quantity $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ has an interpretation as the (incremental) DC gain.
- Our earlier assumption that the plant P is stable (that is, that P_0 is stabilised by C_0) with the extra requirement that P is LES, is sufficient for Condition (ii) to be satisfied.
- Although Condition (iii) nominally requires that $g(h(\xi), \xi)\xi < 0, \forall \xi \neq 0$, if instead it is the case that $g(h(\xi), \xi)\xi > 0, \forall \xi \neq 0$ then we can just change the sign of feedback to achieve closed loop stability. That is, if $g(h(\xi), \xi)\xi \geq \rho|\xi|^2$ for some $\rho > 0$, then there exists a negative value $\epsilon^* < 0$, such that for all $\epsilon^* \leq \epsilon \leq 0$, the equilibrium $(x, \xi) = (0, 0)$ is GAS.
- Condition (iii) may not be satisfied globally. However, if this condition is locally satisfied, then we can instead establish *local* closed loop stability by using Theorem 3.7 to follow.

Note:

If we consider the more general case that the equilibrium point ξ is not zero but fixed at $\xi = \xi_e$ by the influence of a constant reference input r, we require a slight adjustment to Condition (iii). In particular, we require that the equilibrium ξ_e of the reduced model (slow time scale) is GAS and LES (i.e. we should have that $[g(h(\xi))-g(h(\xi_e))](\xi-\xi_e) \leq 0$, and $[g(h(\xi))-g(h(\xi_e))](\xi-\xi_e) \leq \rho|\xi-\xi_e|^2$ is valid for ξ in a neighbourhood of $\xi = \xi_e$.). This will be satisfied for all ξ_e if $\frac{\partial g(h(\xi),\xi)}{\partial \xi} < -\rho < 0$, that is, if the "incremental DC gain" of the nonlinear plant is uniformly bounded away from zero.

We now introduce a theorem from [38] which gives sufficient conditions to guarantee the local stability of a standard singularly perturbed system.

Theorem 3.6 (Conditions for the local stability of a general singular perturbed system)

Consider a nonlinear differential equation

$$\left\{ \begin{array}{lcl} \epsilon \dot{x} & = & f(x,\xi), \quad f : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^n, \\ \dot{\xi} & = & g(x,\xi), \quad g : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^m, \end{array} \right. \tag{3.14}$$

where $f(.,.)$ and $g(.,.)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. Define:

$$A_{11} = \frac{\partial g}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{12} = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)},$$

$$A_{21} = \frac{\partial f}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{22} = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)},$$

and suppose that A_{22} is nonsingular. Suppose further that the solution of the equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (3.13) implicitly defines a C^2 function $x = h(\xi)$. Then the following statements are true.

(i)If all eigenvalues of A_{22} and of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts, there exists an $\epsilon^* > 0$, such that for all $0 < \epsilon < \epsilon^*$, the equilibrium $(x_e = 0, \xi = 0)$ is an asymptotically stable equilibrium point.

(ii)If an eigenvalue of A_{22} or of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ has positive real part, there exists an $\epsilon^* > 0$, such that for all $0 < \epsilon < \epsilon^*$, the equilibrium $(x_e = 0, \xi = 0)$ is an unstable equilibrium point.

Proof The proof is based on the indirect method of Lyapunov and the linear version of the singular perturbation result [38]. ∎

We now specialise the above theorem to the case depicted in Figure 3.3.

Theorem 3.7 (Local conditions for the existence of ϵ^*)

Consider the second case in Figure 3.3 described by equation (3.13), and suppose that $f(.,.)$ and $g(.,.)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. Define:

$$A_{11} = \frac{\partial g}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{12} = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)},$$

$$A_{21} = \frac{\partial f}{\partial \xi}|_{(x,\xi)=(0,0)},$$

$$A_{22} = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)},$$

and suppose that A_{22} is nonsingular. Suppose further that the equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (3.13) has a unique C^2 solution $x = h(\xi)$, and that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonzero. Then

if all eigenvalues of A_{22} and of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts, there exists $\epsilon^* > 0$, such that for all $0 \leq \epsilon \leq \epsilon^*$, the equilibrium $(x_e = 0, \xi_e = 0)$ is an asymptotically stable equilibrium point.

Proof This theorem is a special case of Theorem 3.6. ∎

Remarks:

- Note that if $\dot{x} = f(x,\xi)$ is stable when ξ is fixed, then all the eigenvalues of A_{22} (when ξ is fixed at $\xi = 0$) have negative real parts.

- Note that having $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ nonzero implies that $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is a non zero scalar (See Appendix 3.11 for the proof of a more general case). The sign of the only eigenvalue can therefore be changed by changing the sign of $g(.,.)$. That is, if $A_{11} - A_{12}A_{22}^{-1}A_{21}$ is positive, then closed loop stability can be achieved by changing the feedback from negative to positive. This argument will be generalised to the MIMO case in Section 3.7.

In such a case, it is equivalent to see that if $\dot{x} = f(x, \xi)$ is stable for fixed ξ_e, then there exists $\epsilon^* < 0$, such that for all $\epsilon^* < \epsilon < 0$, the equilibrium (x_e, ξ_e) is locally stable for the closed loop $(P, \frac{\epsilon}{s})$.

3.5 An integrator gain bound

In the last section, we gave sufficient conditions for the existence of a bound on the integrator gain that will guarantee closed loop stability. Here, we will give an explicit expression for such an ϵ^*, based on singular perturbation theory.

Theorem 3.8 (An integrator gain bound)

Consider the second case in Figure 3.3 described by equation (3.13) which satisfies the requirement of equation (3.2), and suppose that the following conditions are satisfied:

(i) There exists a function h such that $x = h(\xi)$ is the unique root of $0 = f(x, \xi)$ in $(x, \xi) \in B_x \times B_\xi$ (Here, B_x and B_ξ are some open balls on x and ξ space respectively).

(ii) There exists a Lyapunov function $W(x, \xi)$ such that for all $(x, \xi) \in B_x \times B_\xi$:

a. $W(x, \xi) > 0$ for all $x \neq h(\xi)$ and $W(h(\xi), \xi) = 0$.

b. There exists some $\alpha_2 > 0$, such that $\frac{\partial W}{\partial x} f(x, \xi) \leq -\alpha_2[\phi(x - h(\xi))]^2$.

c. There exists some γ and β_2 such that $\frac{\partial W}{\partial \xi} g(x, \xi) \leq \gamma[\phi(x - h(\xi))]^2 + \beta_2\psi(\xi)\phi(x - h(\xi))$.

In the above, $\psi(.)$ and $\phi(.)$ are scalar functions of vector arguments which vanish only when their arguments are zero, e.g. $\psi(\xi) = 0$ iff $\xi = 0$.

(iii) There exists a Lyapunov function $V(\xi)$ such that:

d. $\frac{\partial V}{\partial \xi} g(h(\xi), \xi) \leq -\alpha_1\psi^2(\xi)$, for some $\alpha_1 > 0$.

e. There exist some β_1 such that $\frac{\partial V}{\partial \xi}[g(x,\xi)-g(h(\xi),\xi)] \leq \beta_1\psi(\xi)\phi(x-h(\xi))$.

Then, when $0 < \epsilon < \epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma+\beta_1\beta_2}$, there exists a Lyapunov function for the closed loop system $(P, \frac{\epsilon}{s})$ of the form:

$W_\gamma(x,\xi) = (1-d)V(\xi) + dW(x,\xi)$, where d is allowed to be any fixed value in the range $(0,1)$.

Furthermore, the origin is an asymptotically stable equilibrium of $(P, \frac{\epsilon}{s})$.

Proof This theorem is a special case of Theorem 2.1 in page 297 of [58]. ∎

Here, the parameters β_1, β_2 and γ could, in general, be positive, negative or zero. In most problems, however, one arrives at inequalities c and e (in Theorem 3.8) using norm inequalities, leading automatically to nonnegative values for β_1, β_2 and γ [58].

In Section 3.4, sufficient conditions for the existence of an integrator gain bound that guarantees stability were given, and in Theorem 3.8, a value of such an ϵ^* is calculated. However, the relationship between the two theorems is not necessarily obvious. In Theorem 3.9 following, we determine the value of an ϵ^* directly in terms of the parameters of the unaugmented closed loop (P_0, C_0) and the conditions given in Theorem 3.5.

Theorem 3.9 Consider the second case in Figure 3.3 described by equation (3.13) which satisfies the requirement of equation (3.2), and suppose that the following assumptions are satisfied:

(i) f and g are globally Lipschitz continuous with Lipschitz constants k_f and k_g for the Euclidean norm $\|.\|_2$.

(ii) The equation $0 = f(x,\xi)$ obtained by setting $\epsilon = 0$ in equation (3.13) implicitly defines a unique C^2 function $x = h(\xi)$.

(iii) For any fixed $\xi \in \mathcal{R}$ the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x,\xi)$ (that is, the original unaugmented system P) is Globally Exponentially Stable (GES).

(iv) There exists a scalar $\alpha_1 > 0$ such that $g(h(\xi),\xi)\xi \leq -\alpha_1\xi^2, \forall\xi$. This ensures that the equilibrium $\xi = 0$ of the reduced model (slow time scale) $\dot{\xi} = g(h(\xi),\xi)$ is GES.

Then, there exist some $\alpha_2 > 0$, β_1, β_2 and γ such that when $0 < \epsilon < \epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma+\beta_1\beta_2}$, the origin is an asymptotically stable equilibrium of the unforced closed loop $(P, \frac{\epsilon}{s})$.

Proof

For the reduced model (slow time scale), we choose $V(\xi) = \frac{1}{2}\xi^2$, as a Lyapunov function candidate. Then $\frac{\partial V}{\partial \xi} g(h(\xi), \xi) = \xi g(h(\xi), \xi) \leq -\alpha_1 \xi^2$ (This satisfies condition d of Theorem 3.8).

According to the Lipschitz continuity of $g(x, \xi)$, there exists some β_1 such that $\frac{\partial V}{\partial \xi}[g(x, \xi) - g(h(\xi), \xi)] \leq k_g \xi \|x - h(\xi)\| \leq \beta_1 \xi \phi(x - h(\xi))$ (The condition e of Theorem 3.8 is met).

From the condition that for fixed $\xi \in \mathcal{R}$ the equilibrium $x_e = h(\xi)$ of the subsystem $\dot{x} = f(x, \xi)$ is GES, we conclude that there exists a Lyapunov function $W(x, \xi)$ such that

a. $W(x, \xi) > 0 \ \forall x \neq h(\xi)$ and $W(h(\xi), \xi) = 0$.

b. $\frac{\partial W}{\partial x} f(x, \xi) \leq -\alpha_2 \phi^2(x - h(\xi))$ (From the condition that the equilibrium $x_e = h(\xi)$ is GES).

c. $\frac{\partial W}{\partial \xi} g(x, \xi) \leq \gamma[\phi(x - h(\xi))]^2 + \beta_2 \xi \phi(x - h(\xi))$(From the condition that the equilibrium $x_e = h(\xi)$ is GES and the continuity and derivative of $g(.)$ and $h(.)$).

According to Theorem 3.8, we achieve that $\epsilon^* = \frac{\alpha_1\alpha_2}{\alpha_1\gamma+\beta_1\beta_2}$.

∎

3.6 Alternative locations for including an integrator

In the above, we have only discussed the case where the small gain integrator is connected in parallel with the original controller C_0 (See Figure 3.3). Actually, there are also other options for adding such an integrator to the system which will achieve a similar effect. In this section, we discuss alternative options which are depicted in Figures 3.6 and 3.7.

In Figure 3.6, we have started with a stable closed loop system as in Figure 3.2 and have serially connected a transfer function block $\frac{s+\epsilon}{s}$ between the output y_r and

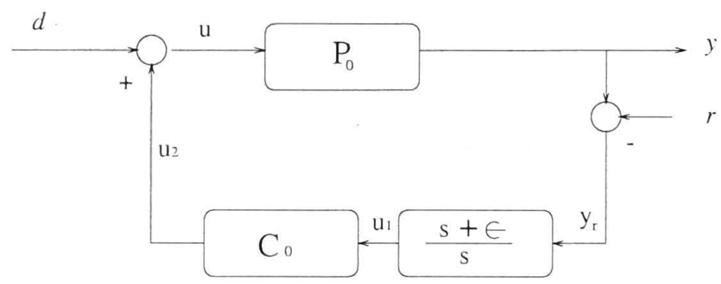

Figure 3.6: Alternative integrator location: Case 3

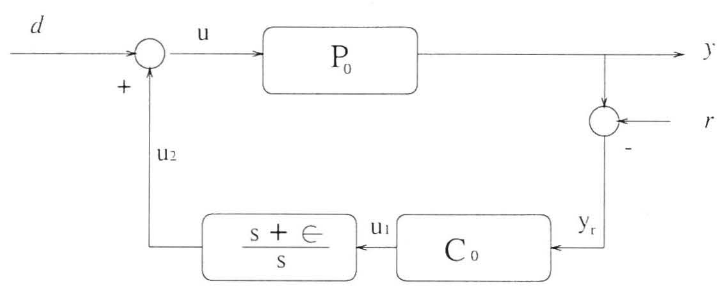

Figure 3.7: Alternative integrator location: Case 4

the original controller C_0.

We will use affine differential equations to simplify our discussion, and therefore assume(with some loss of generality) that we can express P_0, C_0 and $\frac{s+\epsilon}{s}$ as:

$$P_0 : \begin{cases} \dot{x}_0 = f_1(x_0) + f_2(x_0)u \\ y = g_1(x_0) + g_2(x_0)u, \end{cases} \tag{3.15}$$

$$C_0 : \begin{cases} \dot{\eta} = l_1(\eta) + l_2(\eta)u_1 \\ u_2 = m_1(\eta) + m_2(\eta)u_1, \end{cases} \tag{3.16}$$

$$\frac{s+\epsilon}{s} : \begin{cases} \dot{\xi} = \epsilon y_r \\ u_1 = y_r + \xi. \end{cases} \tag{3.17}$$

Here, we assume that $f_1(0) = 0$, $l_1(0) = 0$, $g_1(0) = 0$ and $m_1(0) = 0$.

In order to avoid ill-posedness, it is sufficient to require that $g_2(x_0) = 0$. Then, the equations (3.15) can be simplified as:

$$P_0 : \begin{cases} \dot{x}_0 = f_1(x_0) + f_2(x_0)[u_2 + d] \\ y = g_1(x_0). \end{cases} \tag{3.18}$$

The state equation of the combined system is:

$$Case3: \begin{cases} \dot{x}_0 &= f_1(x_0) + f_2(x_0)[m_1(\eta) + m_2(\eta)(\xi - r + g_1(x)) + d] \\ \dot{\eta} &= l_1(\eta) + l_2(\eta)(\xi - r + g_1(x)) \\ \dot{\xi} &= \epsilon[g_1(x) - r]. \end{cases} \tag{3.19}$$

If we let $x = [x_0^T\ \eta^T]^T$, then the above equation can be rewritten in the form:

$$Case\ 3: \begin{cases} \dot{x} &= \bar{f}_1(x) + \bar{f}_2(x)d + \bar{f}_3(x)[\xi - r + g_1(x)] \\ \dot{\xi} &= \epsilon[g_1(x) - r]. \end{cases} \tag{3.20}$$

Here, $\bar{f}_1(x) = \begin{bmatrix} f_1(x_0) + f_2(x_0)m_1(\eta) \\ l_1(\eta) \end{bmatrix}$, $\bar{f}_2(x) = \begin{bmatrix} f_2(x_0) \\ 0 \end{bmatrix}$,

$\bar{f}_3(x) = \begin{bmatrix} f_2(x_0)m_2(\eta) \\ l_2(\eta) \end{bmatrix}$.

Alternatively let us consider Case 4 as shown in Figure 3.7. See equations (3.15) and (3.16), where we assume that $m_2(\eta) = 0$ in order to avoid ill-posedness. Similarly to case 3, the state equation of the combined system can be written as:

$$Case\ 4: \begin{cases} \dot{x} &= \tilde{f}_1(x) + \tilde{f}_2(x)r + \tilde{f}_3(x)[\xi + d + m_1(x)] \\ \dot{\xi} &= \epsilon m_1(x). \end{cases} \tag{3.21}$$

Here, $\tilde{f}_1(x) = \begin{bmatrix} f_1(x_0) \\ l_1(\eta) + l_2(\eta)g_1(x_0) \end{bmatrix}$, $\tilde{f}_2(x) = \begin{bmatrix} 0 \\ -l_2(\eta) \end{bmatrix}$,

$\tilde{f}_3(x) = \begin{bmatrix} f_2(x_0) \\ l_2(\eta)g_2(x_0) \end{bmatrix}$.

The methods for dealing with cases 3 and 4 are very similar to those for the parallel connection discussed in Sections 3.4 and 3.5. Hence, rather than a full analysis, we just give sufficient conditions for the existence of a scalar ϵ^* such that $0 < \epsilon < \epsilon^*$ guarantees stability of the closed loop. As in case 2, when we analyse the stability of the augmented system, we first neglect the constant input signal r and d [1]. By setting $r = 0$, $d = 0$, equations (3.20) and (3.21) can be analysed both together. We

[1] A nonzero constant reference input r will merely alter the equilibrium state of both P_0 and C_0 as in case 2. A constant disturbance d can also be rejected, if the augmented system is GES.

can choose equation (3.22) following as a common model for cases 3 and 4 to analyse the stability of the augmented system.

$$\begin{cases} \dot{x} &= f_1'(x) + f_3'(x)[\xi + g'(x)] \\ \dot{\xi} &= \epsilon g'(x) \end{cases} \tag{3.22}$$

In the above, f' is the $\bar{f}$ of equation (3.20), or the $\tilde{f}$ of equation (3.21). Similarly, $g'(x)$ is the $g_1(x)$ of equation (3.20), or the $m_1(x)$ of equation (3.21).

In order to use singular perturbation theory, we change equation (3.22) to its standard singular perturbation form.

Let $\tau = \epsilon(t - t_0)$, $\tau = 0$ at $t = t_0$, $\frac{d\tau}{dt} = \epsilon$. It then follows that

$$Cases\ 3,4 : \begin{cases} \epsilon\dot{x} &= f_1'(x) + f_3'(x)[\xi + g'(x)] \\ \dot{\xi} &= g'(x) \end{cases} \tag{3.23}$$

The dot in equation(3.23) means the derivative with respect to τ.

We do not absorb $f_3'(x)g'(x)$ into $f_1'(x)$ in order to emphasise the dependence of the state evolution equation $\dot{x}$ on the plant "output" $g'(x)$.

Theorem 3.10 Consider equation (3.23), which represents either of the augmented systems in Figures 3.6 and 3.7 with $d = 0$ and $r = 0$. Let the following assumptions be satisfied:

(i) The equation $0 = f_1'(x) + f_3'(x)[\xi + g'(x)]$ obtained by setting $\epsilon = 0$ has a unique C^2 solution $x = h(\xi)$.

(ii) For a fixed $\xi \in \mathcal{R}$ the equilibrium $x_e = h(\xi)$ of the subsystem (3.23-1) is Globally Asymptotically Stable (GAS) and Locally Exponentially Stable (LES).

(iii) The equilibrium $\xi = 0$ of the reduced model $\dot{\xi} = [g'(h(\xi))]$ is GAS and LES.

It then follows that there exists an $\epsilon^* > 0$, such that for all $0 < \epsilon < \epsilon^*$, the equilibrium $(x, \xi) = (0, 0)$ is GAS.

Proof The proof is similar to Theorem 3.5. It is also helpful to see p.90 of [60]. ∎

Remarks:

- A sufficient condition for (ii) is that under any constant but arbitrary inputs r and d the closed loop (P_0, C_0) is GAS and LES to some equilibrium $x = x_e$. We now explain why this is the case.

  Note that, if we merely assume that the closed loop (P_0, C_0) is GAS and LES only for zero inputs r and d, then the unperturbed system equation $\dot{x} = f_1'(x) + f_3'(x)[g'(x)]$ is GAS and LES, but we can not ensure that for arbitrary fixed ξ, $\dot{x} = f_1'(x) + f_3'(x)[\xi + g'(x)]$ is also GAS and LES. On the other hand, a fixed arbitrary ξ is equivalent to a constant reference input r (in equation(3.20)) or a constant disturbance input d (in equation(3.21)). Hence, if we assume that the unperturbed system $\dot{x} = f_1'(x) + f_3'(x)[g'(x)]$ is GAS and LES under arbitrary constant inputs r and d then condition (ii) is satisfied.

- A sufficient condition for (iii) may be determined by considering a Lyapunov function candidate $V(\xi) = \frac{1}{2}\xi^2$. We just need that $V(\xi)$ satisfies the following requirements to ensure the reduced model $\dot{\xi} = g'(h(\xi))$ is GAS and LES.

  There exist positive constants $\rho_i, i = 1, 2, 3$ such that

  (a) $\rho_1|\xi|^2 \leq V(\xi) \leq \rho_2|\xi|^2$,

  (b) $\xi\dot{\xi} = g'(h(\xi))\xi \leq -\rho_3|\xi|^2$.

- We can also use methods similar to those in Section 3.5 to give a particular value for such an ϵ^*.

Note:

It also should be emphasised that if we consider the case that the equilibrium point ξ is not zero but fixed at $\xi = \xi_e$ by the influence of a nonzero reference input r, we can add a condition that is more strict than conditions (a) and (b) to ensure the reduced model $\dot{\xi} = g'(h(\xi))$ is GAS and LES for all fixed equilibrium point $\xi = \xi_e$. In particular, for the equilibrium point $\xi = \xi_e$, we assume that $[g'(h(\xi)) - g'(h(\xi_e))](\xi - \xi_e) \leq -\rho_3|\xi - \xi_e|^2$.

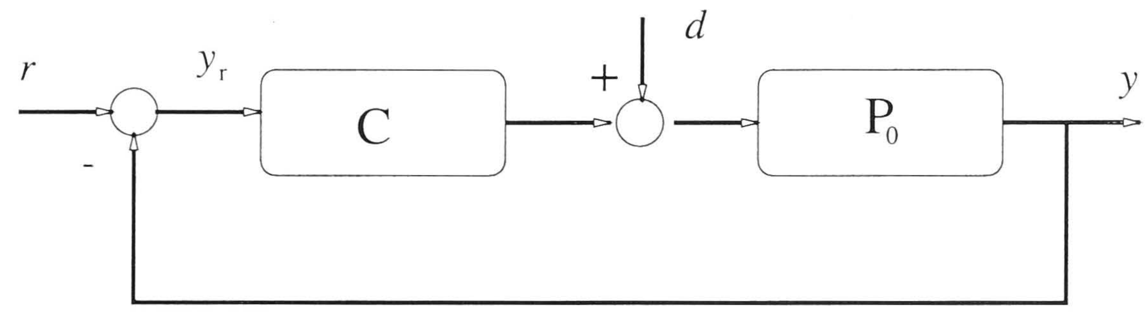

Figure 3.8: The nonlinear MIMO constant disturbance suppression problem

3.7 MIMO systems

So far in the development, we have concentrated our attention on SISO systems. In this section, we will extend our results to MIMO systems as well.

For linear time invariant (LTI) SISO systems, it has been shown that[72] an integrator should be included in the controller to ensure constant input disturbance rejection, regardless of whether the plant itself also has an integrator. For nonlinear MIMO systems, however, it is sometimes complicated to check whether sufficient integrators are included in the controller or plant. We present Theorems 3.11 and 3.12 following to give sufficient conditions for ensuring both constant input disturbance rejection and zero steady state tracking error for nonlinear MIMO systems.

Theorem 3.11 Consider a closed loop system depicted by Figure 3.8 (which may be MIMO system). If one can find or construct a sub-state [2] $x_{itg_i} \in \mathcal{R}$ for each reference tracking error $y_{r_i} \in \mathcal{R}$ (see Figure 3.8) such that

i) $x_{itg_i} = \int \phi_i(y_{r_i})dt$, $i = 1, 2, ..., p$, where, ϕ is a scalar function such that $\phi_i(y_{r_i}) = 0$ iff $y_{r_i} = 0$.

ii)The whole system is stabilised.

Then this closed loop system will reject the constant input disturbance d and ensure zero steady state tracking error for constant reference input.

Proof It is easy to see that if each sub-state x_{itg_i} is stabilised (that is when $t \mapsto +\infty$, x_{itg_i} approaches a constant.), then y_r will go to zero no matter whether the constant input disturbance exist or not. ∎

[2]Given an original state vector of a system satisfying a nonlinear differential equation, a substate of the system is defined as a sub-vector of any Lyapunov transformation of the original state vector.

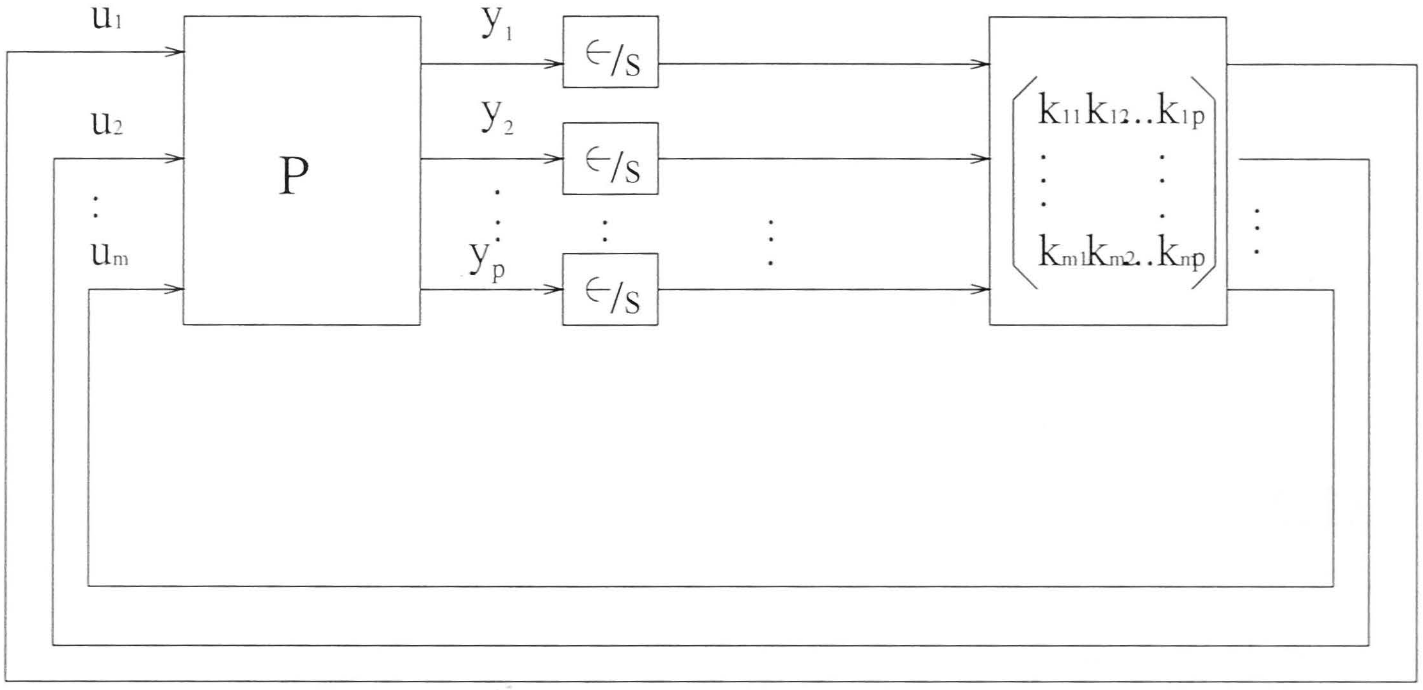

Figure 3.9: MIMO system augmented with small gain integrators

Remarks

- It should be noted that every sub-state x_{itg_i} should be just the integration of a function of the reference tracking error y_{r_i} which vanishes only when its arguments are zero.

- If there already exists such a sub-state in the open loop plant, then it is not necessary to include a small gain integrator $\frac{\epsilon}{s}$ to construct such a sub-state. In the control of a MIMO helicopter model which we present in the next section, the velocity v_y (when ϕ, θ and y_{r_ψ} is small, $\dot{v}_y \approx -\frac{f_{bx}}{m} sin(y_{r_\psi}) \approx -\frac{f_{bx}}{m} y_{r_\psi}$) is just the sub-state of the yaw angle(See Section 3.8). Hence, we have not added $\frac{\epsilon}{s}$ in the yaw angle channel but still acquired constant input disturbance rejection and zero steady state tracking error.

In Theorem 3.11, we have assumed that the MIMO system with the integrators is stable. We have not provided any conditions to ensure stability. In the following, we will give some sufficient conditions to guarantee local stability for a MIMO system augmented with low gain integrators.

As for the SISO analysis, we neglect the constant input signals r and d when we analyse the stability.

Consider Figure 3.9. Let P be a MIMO system with m inputs and p outputs (here, $m \geq p$) described by the following differential equation.

$$\begin{cases} \dot{x} &= f(x,u) \\ y &= g(x,u) \end{cases} \tag{3.24}$$

We assume that $f : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^n$ and $g : \mathcal{R}^n \times \mathcal{R}^m \mapsto \mathcal{R}^p$ are unbiased in the sense that

$$\begin{cases} f(0,0) &= 0 \\ g(0,0) &= 0. \end{cases} \tag{3.25}$$

The state equation of the augmented system can be described as below.

$$\begin{cases} \dot{x} &= f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) \\ \dot{\xi} &= \epsilon g(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i). \end{cases} \tag{3.26}$$

Again, we change equation (3.26) to its standard singular perturbation form.

$$\begin{cases} \epsilon\dot{x} &= f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) \\ \dot{\xi} &= g(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i). \end{cases} \tag{3.27}$$

In equation (3.27), the dot means the derivative with respect to τ.

Theorem 3.12 Consider the system described by equations (3.24) and (3.26) and illustrated in Figure 3.9. Assume that $x = 0$ is an asymptotically stable equilibrium for the plant P, and that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. We assume that

(i) The equation $f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) = 0$ obtained by setting $\epsilon = 0$ in equation (3.27) has a unique C^2 solution $x = h(\xi)$,

(ii) The matrix $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonsingular.

Then, there exists ϵ^* and a constant matrix $K = (k_{ij})_{m\times p}$ (see Figure 3.9) such that $(x = 0, \xi = 0)$ is an asymptotically stable equilibrium whenever $0 < \epsilon < \epsilon^*$.

Proof The proof is in Appendix 3.10. ∎

Synthesis of a controller for global stability is more complicated than that for local stability. However, if a system is globally stable, it is also locally stable. One possible method for designing the constants (k_{ij}) is just to consider local stability, and then perform global stability analysis.

Another practical way to consider the global stability problem is to recursively apply Theorem 3.5 for SISO systems to MIMO systems. Actually, this method can also be applied to deal with the local stability problem (see Theorem 3.12) and achieves a diagonal feedback matrix K when $m = p$.

Specifically, we first view the MIMO system as a SISO system by considering only the input u_1 and output y_1 , that is, we assume that the other inputs are zero and neglect the other outputs. Then, we make the connection of single small gain integrator $\frac{\epsilon}{s}$ and the "SISO" system globally stable if the "SISO" system satisfies the sufficient conditions of Theorem 3.5. Recursively, we connect a second small gain integrator to the augmented "SISO" system with input u_2 and output y_2. Then, the connection of the second single small gain integrator and the augmented "SISO" system is globally stable if the augmented "SISO" satisfies the sufficient conditions of Theorem 3.5, and so on. If the sufficient conditions of Theorem 3.5 are satisfied by each augmented "SISO" system, in this way, we can include all necessary integrators to the MIMO system while ensuring global stability.

3.8 Controller design for a nonlinear helicopter model

In this section, we implement our constant input disturbance rejection method on the simulated control of a helicopter model provided by [41]. In [41], an output tracking controller is designed based on approximate linearisation. However, this method does not completely suppress constant input disturbances. In order to deal with this shortcoming, we augmented an approximate linearised output tracking controller [33] [47] [31] [69], by including an extra integrator block(See Figure 3.10). In Figure 3.10, the numbers near the lines are the dimension of the vectors depicted by the lines.

The helicopter model appears as below in both [41] and [67].

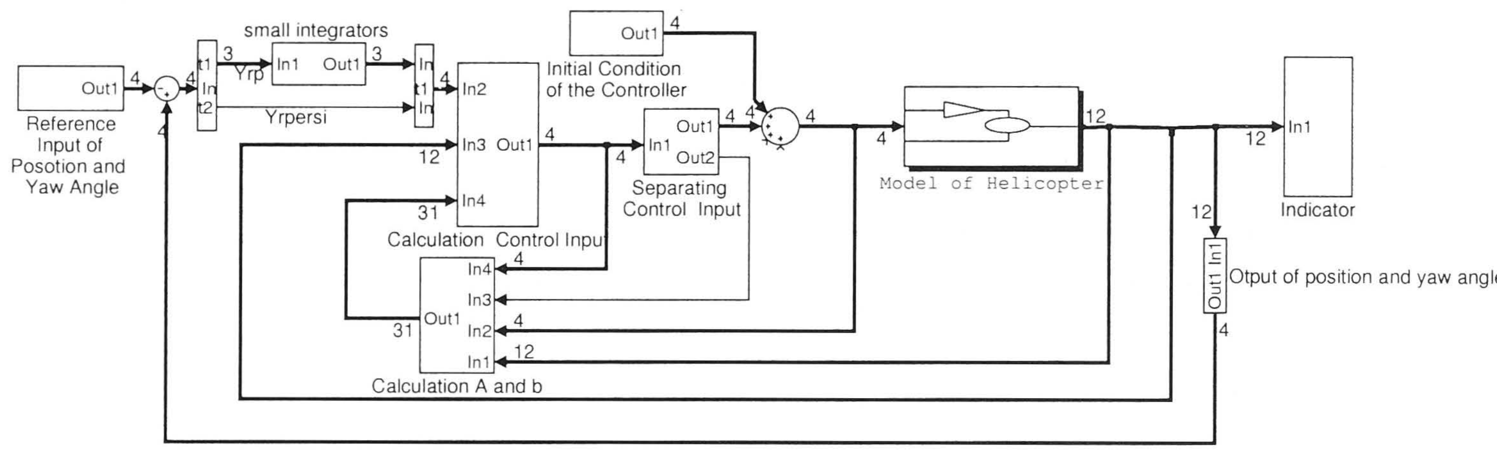

Figure 3.10: Approximate linearised output tracking control with integrator augmentation

$$\begin{bmatrix} \dot{P} \\ \dot{v}^p \\ \dot{R} \\ \dot{\omega}^b \\ \dot{T}_M \\ \dot{T}_T \\ \dot{a}_{1s} \\ \dot{b}_{1s} \end{bmatrix} = \begin{bmatrix} v^p \\ \frac{1}{m} R f^b \\ R\hat{\omega}^b \\ \mathcal{I}^{-1}(\tau^b - \omega^b \times \mathcal{I}\omega^b) \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, \tag{3.28}$$

$$y = [p_x \; p_y \; p_z \; \phi \; \theta \; \psi]^T, \tag{3.29}$$

where $P \in \mathcal{R}^3$ and $v^p \in \mathcal{R}^3$ are the position and velocity vectors of centre of the mass in spatial coordinates, $R \in \mathcal{SO}(3)$ is the rotation matrix of the body axes relative to the partial axes, $w^b = [w_x^b, w_y^b, w_z^b]^T \in \mathcal{R}^3$ is the body angular velocity vector, $\hat{w}^b \in \mathcal{R}^{3\times3}$ is defined as in equation (3.30), $m > 0$ is the body mass, $\mathcal{I} \in \mathcal{R}^{3\times3}$ is the inertial matrix and $f^b, \tau^b \in \mathcal{R}^3$ are the body force and torque.

$$\hat{w}^b = \begin{bmatrix} 0 & -w_z^b & w_y^b \\ w_z^b & 0 & -w_x^b \\ -w_y^b & w_x^b & 0 \end{bmatrix} \tag{3.30}$$

The body forces and torques generated by the main rotor are controlled by T_M,a_{1s} and b_{1s}, in which a_{1s} and b_{1s} are respectively the longitudinal and lateral tilt of the tip path plane of the main rotor with respect to the shaft. The tail rotor is considered as the source of pure lateral force and anti-torque, which are controlled by T_T.

As in [41], we also assume that all the states are measurable. In order to present the helicopter system in an input-affine form, we define $w = [w_1\ w_2\ w_3\ w_4]^T$, which are the derivatives of T_M, T_T, a_{1s} and b_{1s} as auxiliary inputs to the system. Here the state $x \in \mathcal{R}^{16}$, the inputs $w \in \mathcal{R}^4$, the output $y \in \mathcal{R}^6$.

It can be seen that the helicopter model (3.28) is marginally unstable, so we first need to design a stabilising controller. As in [41], we can design an approximate linearised output tracking controller. Based on this controller we then design a modified controller by augmentation with integrators, and achieve satisfactory disturbance rejection results.

The system equations of (3.28) and (3.29) have four control inputs so the maximum number of outputs for possibly applying an input-output linearisation procedure is four. We choose the outputs p_x, p_y, p_z, ψ as in [41].

Approximate linearisation is implemented by neglecting the coupling terms, a procedure which is presented very clearly in [41].

We define reference tracking error signals as follows:

$$y_{r_{p_i}} = r_{p_i} - y_{p_i},$$

$$y_{r_\psi} = r_\psi - y_\psi.$$

Here, $i = x, y, z$, while r_{p_i} and r_ψ are the reference inputs for position and yaw angle respectively.

We augment the approximate input-output linearisation controller with the small gain integrators of the reference tracking error of position $y_{r_{px}}, y_{r_{py}}$ and $y_{r_{pz}}$ (see Figure 3.10). We define the output of the small gain integrators as

$$u_{p_i} = \frac{k_{p_i}}{s} y_{r_{p_i}}$$

According to Theorem 3.12, it is possible to choose values for k_{p_i} to retain the stability of the system while acquiring constant input disturbance rejection and zero steady state tracking error.

As we stated in last section, we have not integrated the yaw angle reference tracking error y_{r_ψ}, because when ϕ, θ and ψ is small, $\dot{v}_y \approx -\frac{f_{bx}}{m} sin(y_{r_\psi}) \approx -\frac{f_{bx}}{m} y_{r_\psi}$. That is v_y is just the sub-state x_{itg_ψ} corresponding to the yaw angle reference tracking error y_{r_ψ} that is required in Theorem 3.11. So, we need not augment the controller with an integrator for y_{r_ψ}.

Figures 3.11 and 3.12 illustrate some simulation results, where the mass of the helicopter has changed 10 percent from the nominal case. Although a 10 percent change in mass is, strictly speaking, a change in the plant rather than a constant input disturbance, it has a similar effect of altering the (constant) control input required to achieve equilibrium. We consider such a "disturbance" in order to make comparison with the results reported in [41]. The augmented controller is still able to track the reference input without steady state errors. In contrast, the approximate input-output linearisation controller without the integrator augmentation does not reject such disturbances.

3.9 Conclusion

In this chapter we have addressed the problem of achieving constant input disturbance rejection and constant reference tracking, for nonlinear systems. A relatively intuitive solution to this problem has been proposed: we simply augment an existing controller (which stabilises the nonlinear system) with (an) appropriately located integrator(s), with appropriately small gain. We can use singular perturbation theory to guarantee that, even with the addition of such an integrator, closed loop stability will be retained. It is also straightforward to deduce that the inclusion of an appropriately located integrator in the closed loop will ensure that constant input disturbances are in fact, rejected and that constant references will be tracked.

A performance tradeoff with respect to the integrator gain certainly holds for linear systems. We expect that such a tradeoff would also hold in general. This tradeoff is between the time constant associated with the suppression of the constant signals and the performance with respect to other disturbance signals for which the original controller was designed. The speed of the constant suppression (the slow time-scale system), in general, increases with the magnitude of the integrator gain. However, as the magnitude of this gain increases, the closed-loop performance is no

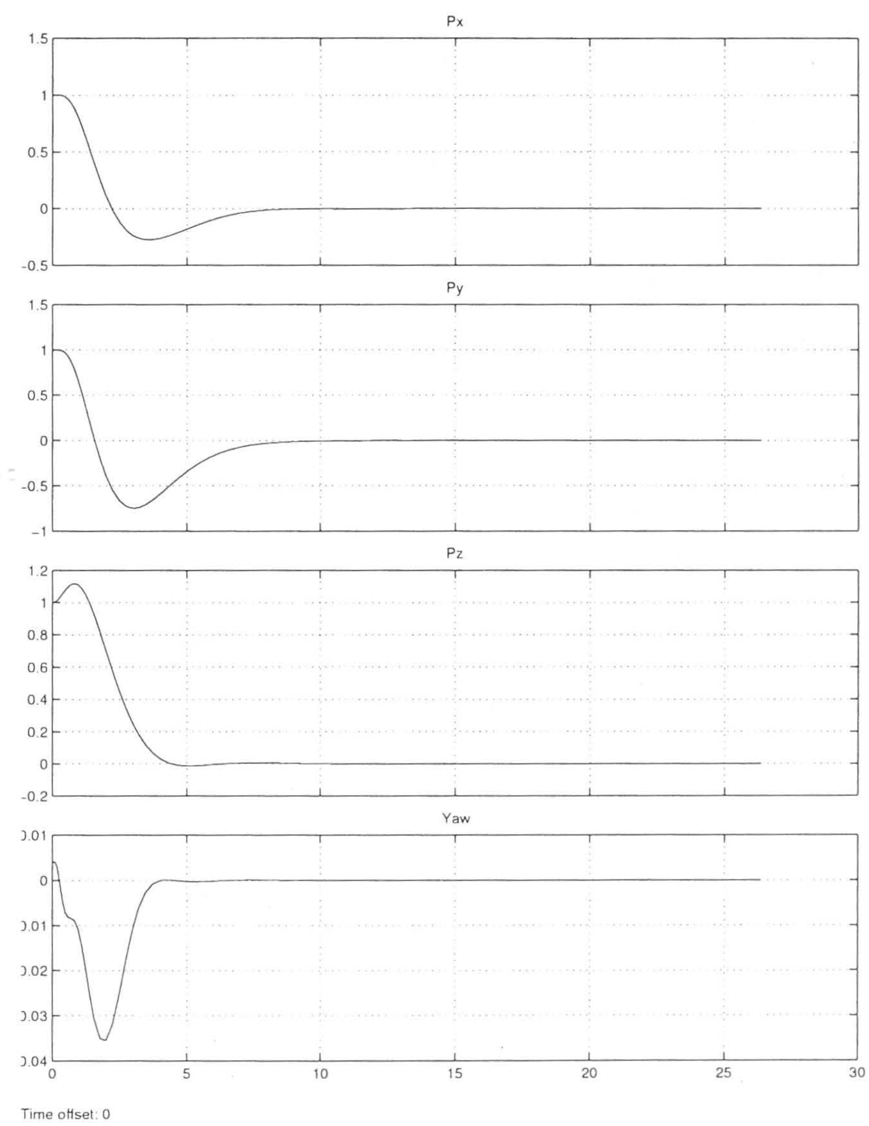

Figure 3.11: The output of use augmented system under 10 percent change of mass

longer guaranteed by singular perturbation theory to accurately approximate the ideal two time-scale system, and the closed loop may approach instability, yielding poor performance for some classes of disturbances.

Our simulation results on a nonlinear helicopter model indicate that satisfactory performance can be achieved in some circumstances, and that the proposed method is a simple but effective way to achieve the suppression of exogenous signals.

3.10 Proof of Theorem 3.12

Proof For augmented system (3.26), we can apply Theorem 3.6 by making the following matrix identifications.

$A_{11} = G_u K,$

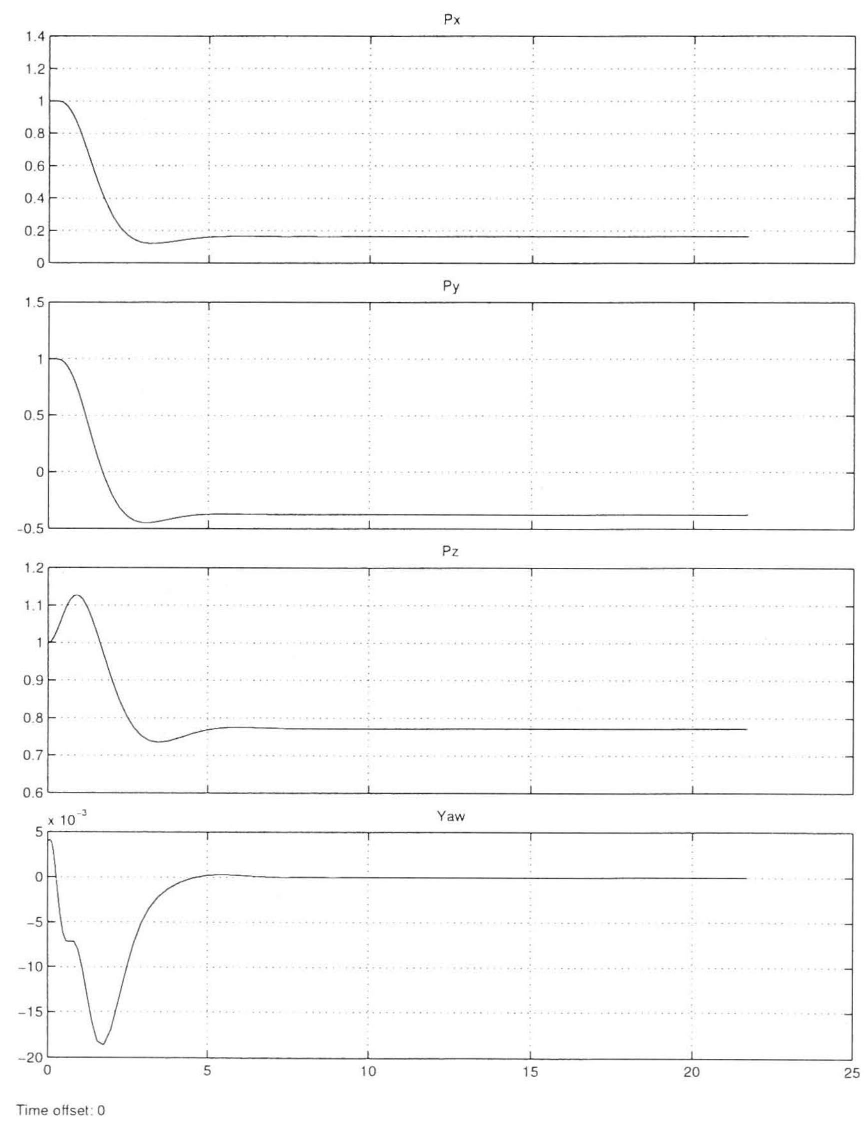

Figure 3.12: The system Output without the augmentation under 10 percent change of Mass

$A_{12} = G_x,$

$A_{21} = F_u K,$

$A_{22} = F_x.$

Here,

$$G_u = \frac{\partial g}{\partial u}|_{(x,\xi)=(0,0)} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \frac{\partial g_1}{\partial u_2} & \cdots & \frac{\partial g_1}{\partial u_m} \\ \frac{\partial g_2}{\partial u_1} & \frac{\partial g_2}{\partial u_2} & \cdots & \frac{\partial g_2}{\partial u_m} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial g_p}{\partial u_1} & \frac{\partial g_p}{\partial u_2} & \cdots & \frac{\partial g_p}{\partial u_m} \end{bmatrix}_{(x,\xi)=(0,0)},$$

$$K = \begin{bmatrix} k_{11} & k_{12} & \dots & k_{1p} \\ k_{21} & k_{22} & \dots & k_{2p} \\ \dots & \dots & \dots & \dots \\ k_{m1} & k_{m2} & \dots & k_{mp} \end{bmatrix},$$

$$G_x = \frac{\partial g}{\partial x}|_{(x,\xi)=(0,0)} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \dots & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & \dots & \frac{\partial g_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial g_p}{\partial x_1} & \frac{\partial g_p}{\partial x_2} & \dots & \frac{\partial g_p}{\partial x_n} \end{bmatrix}_{(x,\xi)=(0,0)},$$

$$F_u = \frac{\partial f}{\partial u}|_{(x,\xi)=(0,0)} \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \dots & \frac{\partial f_1}{\partial u_n} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \dots & \frac{\partial f_2}{\partial u_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \dots & \frac{\partial f_n}{\partial u_n} \end{bmatrix}_{(x,\xi)=(0,0)},$$

$$F_x = \frac{\partial f}{\partial x}|_{(x,\xi)=(0,0)} \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{(x,\xi)=(0,0)}.$$

As we have assumed that $x = 0$ is an asymptotically stable equilibrium for plant P, the eigenvalues of A_{22} have negative real parts.

From the assumption that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonsingular, we can conclude that $(G_u - A_{12}A_{22}^{-1}F_u)K$ is also nonsingular (See Appendix 3.11). This implies that $(G_u - A_{12}A_{22}^{-1}F_u)$ is full row rank.

Because $A_{11} - A_{12}A_{22}^{-1}A_{21} = (G_u - A_{12}A_{22}^{-1}F_u)K$, we can choose a K to ensure that all eigenvalues of $A_{11} - A_{12}A_{22}^{-1}A_{21}$ have negative real parts.

According to Theorem 3.6, there exists ϵ^* and a matrix $K_{m\times p}$ such that $(x = 0, \xi = 0)$ is an asymptotically stable equilibrium whenever $0 < \epsilon < \epsilon^*$.

Furthermore, when $m = p$, the stabilising control matrix K can be chosen to be triangular or even diagonal (See Appendix 3.12).

∎

3.11 Proof of Non-singularity

Lemma 3.13 Consider the system described by equations (3.24) and (3.27) and illustrated in Figure 3.9. Assume that $x = 0$ is an asymptotically stable equilibrium for the plant P, and that $f(\cdot,\cdot)$, $g(\cdot,\cdot)$ are continuously differentiable with $f(0,0) = 0$ and $g(0,0) = 0$. We assume that

(i) The equation $f(x, \sum_{i=1}^{p} k_{1i}\xi_i, \sum_{i=1}^{p} k_{2i}\xi_i, ..., \sum_{i=1}^{p} k_{mi}\xi_i) = 0$ obtained by setting $\epsilon = 0$ in equation (3.27) has a unique C^2 solution $x = h(\xi)$,

(ii) The matrix $\frac{\partial g(h(\xi),\xi)}{\partial \xi}|_{\xi=0}$ is nonsingular.

Then, $(G_u - A_{12}A_{22}^{-1}F_u)K$ (G_u, A_{12}, A_{22} and F_u are defined as in Theorem 3.12) is nonsingular.

Proof Consider that $f(h(\xi),\xi) = 0$, we have

$\frac{\partial f}{\partial x}\frac{\partial h}{\partial \xi} + \frac{\partial f}{\partial u}\frac{\partial u}{\partial \xi} = 0.$

That is

$F_x\frac{\partial h}{\partial \xi} + F_u K = 0$, so

$\frac{\partial h}{\partial \xi} = -F_x^{-1}F_u K.$

Now, we have

$\frac{\partial g(h(\xi),\xi)}{\partial \xi} = \frac{\partial g}{\partial x}\frac{\partial h}{\partial \xi} + \frac{\partial g}{\partial u}\frac{\partial u}{\partial \xi} = (G_u - A_{12}A_{22}^{-1}F_u)K.$

From the assumption that $\frac{\partial g(h(\xi),\xi)}{\partial \xi}$ is nonsingular, we conclude that $(G_u - A_{12}A_{22}^{-1}F_u)K$ is also nonsingular.

■

3.12 Proof of the Existence of a Stabilising Diagonal Matrix K

Lemma 3.14 Define A_i, the $i \times i$ (upper left) sub-matrix of A as

$$A_i = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1i} \\ a_{21} & a_{22} & \dots & a_{2i} \\ \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ii} \end{bmatrix}. \tag{3.31}$$

For any nonsingular matrix $A_0 \in \mathcal{R}^{n \times n}$, it is possible to reorder the columns of A_0 to ensure the reordered matrix $A = (a_{ij}) \in \mathcal{R}^{n \times n}$ satisfies the property that for each $i = 1, 2, ...n$, $det(A_i) \neq 0$.

Proof It is easy to see this if we consider each column as a vector. ■

Lemma 3.15 If all the roots of the equation $s^n + a_1 s^{n-1} + ... + a_0 = 0$ have negative real parts, then there exists an ϵ^* such that when $0 < \epsilon < \epsilon^*$ all the roots of the equation $s^{n+1} + a_1 s^n + ... + a_0 s + \epsilon = 0$ have negative real parts.

Proof This is a direct result of linear singular perturbation theory. ■

Theorem 3.16 For a nonsingular square matrix $A \in \mathcal{R}^{n \times n}$ with A_i defined as in (3.31) and $det(A_i) \neq 0$ for each $i = 1, 2, ...n$ there exists a diagonal matrix $K \in \mathcal{R}^{n \times n}$ such that all the eigenvalues of the matrix AK have negative real parts

Proof Write K as

$$K = \begin{bmatrix} k_1 & 0 & \dots & 0 \\ 0 & k_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & k_n \end{bmatrix}. \tag{3.32}$$

Then,

$$AK = \begin{bmatrix} k_1 a_{11} & k_2 a_{12} & \dots & k_n a_{1n} \\ k_1 a_{21} & k_2 a_{22} & \dots & k_n a_{2n} \\ \dots & \dots & \dots & \dots \\ k_1 a_{n1} & k_2 a_{n2} & \dots & k_n a_{nn} \end{bmatrix}, \tag{3.33}$$

$$
\begin{aligned}
& det\left([sI - AK]\right) \\
= \quad & s^n + (k_1 a_{11} + k_2 a_{22} ... + k_n a_{nn}) s^{n-1} \\
+ \quad & (k_1 k_2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + k_1 k_3 \begin{bmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{bmatrix} + ... k_{n-1} k_n \begin{bmatrix} a_{n-1\ n-1} & a_{n-1\ n} \\ a_{n\ n-1} & a_{n\ n} \end{bmatrix}) s^{n-2} \\
+ \quad & (k_1 k_2 k_3 \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} + ... + ... k_{n-2} k_{n-1} k_n \begin{bmatrix} a_{n-2\ n-2} & a_{n-2\ n-1} & a_{n-2\ n} \\ a_{n-1\ n-2} & a_{n-1\ n-1} & a_{n-1\ n} \\ a_{n\ n-2} & a_{n\ n-1} & a_{n\ n} \end{bmatrix}) s^{n-3} \\
+ \quad & ... \\
+ \quad & k_1 k_2 ... k_n \begin{bmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & a_{nn} \end{bmatrix}
\end{aligned}
\tag{3.34}
$$

Now, we set $|\frac{k_2}{k_1}| = \varepsilon_1$, $|\frac{k_3}{k_2}| = \varepsilon_2$, ... $|\frac{k_n}{k_{n-1}}| = \varepsilon_{n-1}$, and

$\bar{\varepsilon} = max\{\varepsilon_1, \varepsilon_2, ...\varepsilon_{n-1}\}$.

Let $\bar{\varepsilon} \ll 1$, then

$$
\begin{aligned}
& det\left([sI - AK]\right) \\
= \quad & s^n + (k_1 det(A_1) + \mathcal{O}(\varepsilon_1)) s^{n-1} \\
+ \quad & (k_1 k_2 det(A_2) + \mathcal{O}(\varepsilon_1 \varepsilon_2)) s^{n-2} \\
+ \quad & (k_1 k_2 k_3 det(A_3) + \mathcal{O}(\varepsilon_1 \varepsilon_2 \varepsilon_3)) s^{n-3} \\
+ \quad & ... \\
+ \quad & k_1 k_2 k_3 ... k_n det(A_n)
\end{aligned}
\tag{3.35}
$$

Consider that $det(A_i) \neq 0$, for all $i = 1, 2, ...n$ and apply lemma 3.15 repeatly. Firstly, we choose a value for k_1 and a small enough ε_1^* such that when $0 < \varepsilon_1 < \varepsilon_1^*$ all the roots of $s + (k_1 det(A_1) + \mathcal{O}(\varepsilon_1)) = 0$ have negative real parts. Secondly, we choose a value for k_2 (after the choice of k_2, ε_1 can be fixed) and a small enough ε_2^* such that when $0 < \varepsilon_2 < \varepsilon_2^*$ all the roots of $s^2 + (k_1 det(A_1) + \mathcal{O}(\varepsilon_1)) s + (k_1 k_2 det(A_2) + \mathcal{O}(\varepsilon_1 \varepsilon_2)) =$

0 have negative real parts. We continue this procedure. Finally, we choose a small enough k_n such that all the roots of the nth-order equation have negative real parts.

∎

Chapter 4

Robust disturbance suppression for nonlinear systems based on multiple model adaptive control

In this chapter, we will deal with the disturbance suppression problem for nonlinear systems based on Multiple Model Adaptive Control (MMAC). In the first section, we present a method to construct a stable multi-estimator for an open-loop unstable nonlinear plant based on the concept of a stable kernel representation [74]. We also provide an example to show the design of the multi-estimator and multi-controller to ensure constant disturbance rejection as well as constant reference tracking under plant variation. In the second section, in order to make MMAC with its use of multi-estimators and multi-controllers more efficient and practical, we probe efficient ways of multi-realisation for multi-controller and multi-estimator structure, named minimal (and minimal "generic") stably based multi-realisation. We provide the necessary and sufficient conditions for the multi-realisation of a family of linear multi-variable systems based on matrix fractional descriptions. Furthermore, we introduce the new concept of hc-dependence, and provide the necessary and sufficient conditions for hc-dependence. Finally, the minimal (and minimal "generic") stably based multi-realisation problems are solved for linear MIMO systems based on hc-dependence [10] [71]. Although we have not presented a comprehensive theory for multi-controllers and multi-estimators for nonlinear systems (in contrast to an example demonstrating their feasibility), we have constructed part of the basis of such a theory, in the consideration of MMAC for MIMO linear time invariant systems.

4.1 Robust input disturbance suppression for nonlinear systems based on multiple model adaptive control

In this section, we make initial steps in the direction of nonlinear multiple model adaptive control by focusing on a contrived example in which an unknown parameter has a nominal value in one of the two intervals $[-0.3, -0.1]$ and $[0.1, 0.3]$ and can switch between them. The design should suppress a constant input disturbance. We discuss the use of a multi-estimator and multi-controller to achieve the goal, with the construction of the stable multi-estimator being based on stable kernel representation of the plant. Simulation results indicate that satisfactory performance is achieved.

4.1.1 Introduction

This section serves to marry two ideas: controller design to secure constant input disturbance rejection for a nonlinear system, and multiple model adaptive control. To illustrate the ideas, we shall work with an underlying nonlinear plant, containing a parameter which can take values in one of two non-overlapping intervals (and the parameter can switch intervals but not extremely frequently). Robust control design is required, with an adaptive overlay, taken here to be based on multiple model adaptive control.

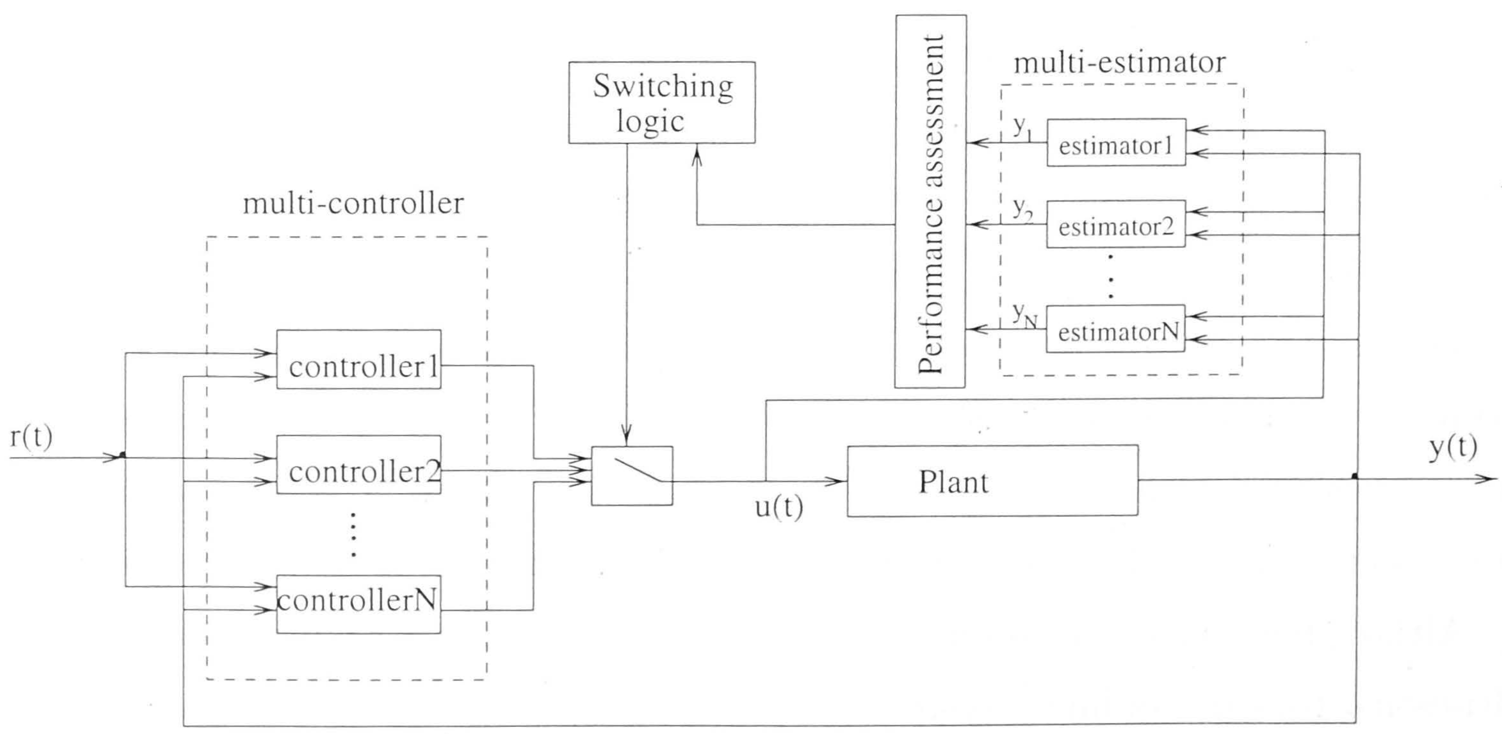

Figure 4.1: A Multiple Model Adaptive Control system.

MMAC is a model-based control strategy which incorporates a set of model/controller pairs rather than relying on a single model and controller to handle all possible operating conditions (see Figure 4.1). More precisely, MMAC algorithms assume that the unknown true plant either belongs to a given finite set of nominal plants, or is at least in some way close to one (or more) members of that set [5]. Each nominal plant corresponds to a controller that is presumed to give satisfactory performance in conjunction with both the nominal plant, and an associated uncertainty ball [5].

A number of excellent text and monographs ([4], [5], [29], [52], [54] and [57]) have been written in the area of MMAC, especially for linear plants. Paper [54] actually provides a way to achieve robust (constant) disturbance suppression and constant reference tracking for a linear SISO plant based on supervised control system. The main disturbance suppression methodology, not unsurprisingly, is to integrate the reference tracking error by including an integrator in the controller. It has been shown in [54] that the supervisory part of the controller can orchestrate the switching of a sequence of candidate controllers into feedback with the system so as (i) to cause the output of the process to approach and track a constant reference input despite norm-bounded unmodelled dynamics, and constant process disturbances and (ii) to ensure that none of the signals within the overall system can grow without bound in response to bounded disturbances, be they constant or not.

This section is a first step in the direction of extending some of these ideas to nonlinear plants. The notion of constant disturbance suppression for nonlinear systems is reasonably straightforward, see [19], [70] and [72]. The key issue is to explore how to achieve an MMAC capability, and this is in turn rests on having a so-called stable multi-estimator. The stable multi-estimator for a possibly unstable nonlinear plant is constructed based on stable kernel representations. This is one way an extension of the linear system ideas in the papers [5] and [53] can be achieved.

Many problems related to the extension of papers [4], [5], [29], [52] and [54] to the nonlinear case are still open, such as the choice of nominal models, robustness analysis, securing of safe switching, boundness analysis for the disturbance - to - tracking - error gain, transient response (dwell-time switching) and so on.

4.1.2 Multi-estimator design for nonlinear plants

In this subsection, we first recall briefly the multi-estimator design method for linear plants introduced in [5]. Then this method is directly extended to nonlinear plants, based on the concept of normalised stable kernel representation, see [20] and [63].

Multi-estimator design for linear plants

In this subsection, the multi-estimator (a main part of the supervisor in an MMAC scheme) for linear SISO plants is briefly introduced, based on [5].

The main task of the multi-estimator is plant identification, or more precisely, hypothesis testing. A true (unknown) plant P exists, together with a collection $\{P_1, P_2, \cdots, P_N\}$ of nominal plants. For each nominal plant P_i, there is an assumed hypothesis H_i, that the true plant lies in an uncertainty ball around P_i [5], and is closer to P_i than P_j for any $j \neq i$ (How one measures closeness is also an issue). The multi-estimator (together with the Performance Assessment and Switching Logic blocks in Figure 4.1) determines the most likely hypothesis, and switches in the corresponding controller C_i. Of course, C_i is chosen to be a good controller for P_i and indeed plants in an uncertainty ball around P_i. Just how many P_i are needed and where they should be located are issues which have only very recently been addressed in the linear case, see [4], and they remain as pertinent issues for the nonlinear problem. Other factors affecting the hypothesis testing include the effects of noise, errors rates in the hypothesis testing scheme, the time required to make a decision and so on.

For a linear plant (see Figure 4.1), the multi-estimator is a linear system driven by the unknown plant input u and output y, with N outputs $y_1, y_2, \cdots, y_N$. They have the special property that if $P = P_i$, then $y = y_i$ (after the decay of initial condition effects, in the absence of noise, and given that all signals are bounded). The error signals $y - y_j$, $j = 1, 2, \cdots N$, are used to determine which P_i is closest to P. Usually, an exponentially weighted $\mathcal{L}_2$ norm is need to compare the errors. Even with these specifications, there is still much freedom in the design of the multi-estimator [5].

The structure of the multi-estimator presented in [5] is as follows. For N nominal models P_i, we identify their transfer functions $P_i = \frac{n_i(s)}{d_i(s)}$, with $n_i(s)$ and $d_i(s)$ coprime polynomials. For a stable polynomial $D(s)$, the part of the multi-estimator linking

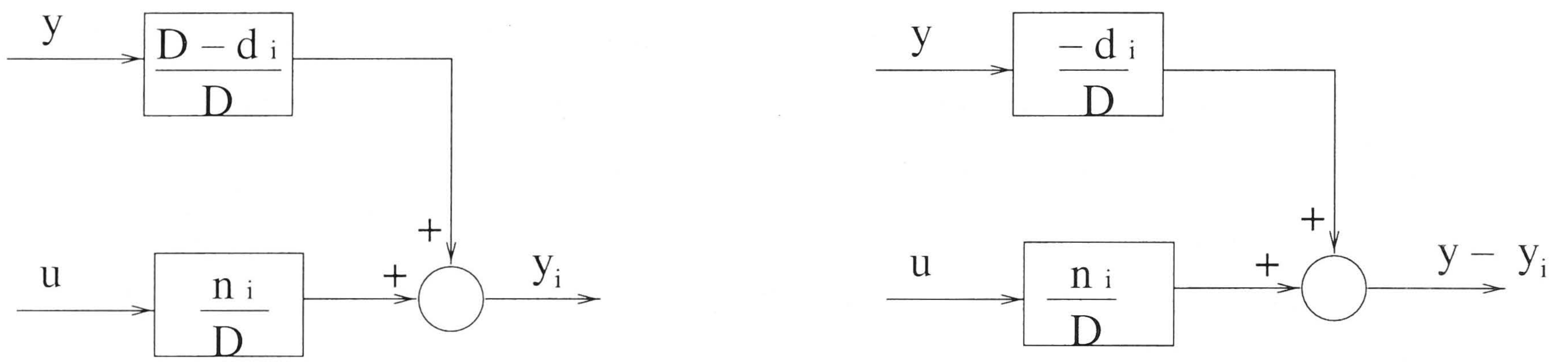

Figure 4.2: Constituent part of multi-estimator.

$[y, u]^T$ to y_i is depicted in the left hand side of Figure 4.2. Note that if the transfer function of P were equal to that of the nominal plant model P_i, then, in the absence of noise and disturbances and given bounded signals, y_i would converge to y asymptotically (This is true even if P is open loop unstable). The use of the same D for all i allows much simplification in the internal construction (state variable realisation) of the multi-estimator. However, in order to ensure the error signals are comparable for different P_i, it appears more logical to use different normalising stable polynomials $D_i(s)$ instead of a single stable polynomial $D(s)$. That is, $D_i(s)$ satisfies

$$D_i^*(s)D_i(s) = n_i^*(s)n_i(s) + d_i^*(s)d_i(s). \tag{4.1}$$

The multi-estimator in effect is then N separate systems, with common input vector $(u, y)^T$

Multi-estimator design for nonlinear plants

In this subsection, the linear multi-estimator will be extended to nonlinear case. Although there is not a comprehensive analysis for this extended nonlinear multi-estimator, it still possesses the special property that if $P = P_i$, then $y = y_i$ (after decay of initial condition effects, in the absence of noise, and given that all signals are bounded) as in the linear case.

For the linear multi-estimator depicted in the left hand side of Figure 4.2, set $z_i = y - y_i$, so that

$$z = [-\frac{d_i(s)}{D_i(s)} \vdots \frac{n_i(s)}{D_i(s)}][y\, u]^T. \tag{4.2}$$

If $[y, u]^T$ is actually the output and input of $P_i(s)$, i.e. if $P(s) = P_i(s)$, then

$z = 0$. This is equivalent to saying that the multi-estimator implements a stable kernel representation of the system corresponding to the nominal model P_i: thus $[-\frac{d_i(s)}{D_i(s)} \vdots \frac{n_i(s)}{D_i(s)}]$ is a stable transfer matrix with "inputs" y and u, and "output" z, while the input-output behaviour of the system corresponding to P_i is the set of all pairs (y, u) which are mapped by $[-\frac{d_i(s)}{D_i(s)} \vdots \frac{n_i(s)}{D_i(s)}]$ onto $z = 0$. Therefore, if the nominal model P_i exactly matches the plant P, then the multi-estimator maps the pair (y, u), which is corresponding to the true plant, onto $z = 0$. Because $D_i(s)$ satisfies equation (4.1), this multi-estimator actually is a normalised stable kernel representation.

Now let us extend this to the nonlinear case, based on ideas of [20] and [63]. For simplicity we consider only a nominal plant P_i that is affine in the control:

$$P_i \left\{ \begin{array}{rcll} \dot{x} & = & f(x) + g(x)u, & x \in \mathcal{R}^n, u \in \mathcal{R}^m, \\ y & = & h(x), & y \in \mathcal{R}^p. \end{array} \right. \tag{4.3}$$

Here, $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ are sufficiently smooth functions ensuring that a well-defined response exists for all $u(\cdot)$ in some suitable class (certainly containing piecewise continuous, but not necessarily bounded functions), and $f(0) = 0$, $h(0) = 0$. It is possible to construct a particular normalised stable kernel representation of some classes of nonlinear plant, provided there exists a solution to a certain partial differential equation.

Theorem 4.1 Consider equation (4.3). Assume there exists a scalar solution $W(x) \geq 0$ to the following Hamilton-Jacobi equation

$$W_x(x)f(x) + \frac{1}{2}W_x(x)g(x)g^T(x)W_x^T(x) - \frac{1}{2}h^T(x)h(x) = 0, \tag{4.4}$$

and a solution $k(x)$ to

$$W_x(x)k(x) = h^T(x).$$

Then the system

$$K_{P_i} \left\{ \begin{array}{rcl} \dot{x} & = & [f(x) - k(x)h(x)] + g(x)u + k(x)y \\ z & = & y - h(x), \end{array} \right. \tag{4.5}$$

is a normalised stable kernel representation of P_i which has $\mathcal{L}_2$-gain equal 1.

Proof See [20]. ∎

To assist the reader, we note that in the linear case, a certain co-inner transfer function matrix is defined by (4.5), and this explains the word "normalised". The stability property means that if $x(0) = 0$, then $u(\cdot)$, $y(\cdot) \in \mathcal{L}_2[0,\infty)$ imply $z \in \mathcal{L}_2[0,\infty)$, and the kernel property means that if $z \equiv 0$, then u, y correspond to some input/output pair of P_i, and vice versa.

Theorem 4.1 can be directly used to construct a multi-estimator for a collection of nonlinear nominal models P_i, $i = 1, 2, \cdots, N$. Even for an unstable nonlinear nominal model P_i, the corresponding multi-estimator is also stable and has $\mathcal{L}_2$-gain equal 1. Note however that (4.4) is not necessarily solvable for all $f(\cdot)$, $g(\cdot)$ and $h(\cdot)$ tuples.

For performance assessment (see Figure 4.1), we still look at an (exponentially weighted) $\mathcal{L}_2$ norm [5] as in the linear case. More precisely, the switching logic relies on a particular choice of monitoring signals defined as follows:

$$\mu_i = \int_0^t e^{-\lambda(t-\tau)} z_i^2 d\tau. \tag{4.6}$$

where λ is a positive smoothing constant.

It is not straightforward to give rules for the selection of λ. However, we would expect to smooth over a longer time than the time constant of the natural dynamics of the P_i, and over a shorter time than the typical interval over which P can change substantially, should it be time-varying.

4.1.3 An Example

In this subsection, a simple admittedly academic example of MMAC for the nonlinear plant is presented to highlight some fundamental issues of the robust disturbance suppression for the nonlinear case.

The nonlinear plant model is given as follows.

$$P \begin{cases} \dot{x} &= \theta x^3 + u \\ y &= x^3, \end{cases} \tag{4.7}$$

where θ is a piecewise constant. Its nominal value is -0.2 ± 0.1, but it can suddenly change its sign to 0.2 ± 0.1. Sign changes are assumed to be infrequent, in the sense that after a change, the adaptive system should be able to learn that change before the next one. Assume two nominal models are given as the following form.

$$P \begin{cases} \dot{\xi} &= \theta_i \xi^3 + u \\ y &= \xi^3, \end{cases} \tag{4.8}$$

where $i = 1, 2$, $\theta_1 = 0.22$ and $\theta_2 = -0.18$.

The aim of the control design is to ensure constant reference input tracking under constant input disturbance, together with an ability to handle changes in P.

For constant reference input tracking under a constant input disturbance, we can augment any stabilising design with a small gain integrator to achieve this goal (see [70]). To handle adaptivity, we build on ideas of [54].

The first thing is to design the multi-estimator. In the last subsection, a method to construct a nonlinear multi-estimator is presented, which involves solution of the Hamilton-Jacobi equation (4.4).

A semi-positive solution of the Hamilton-Jacobi equation (4.4) for nominal model is achieved as follows:

$$W(\xi) = \frac{1}{4}(\sqrt{1+\theta_i^2} - \theta_i)\xi^4.$$

with

$$\frac{\partial W(\xi)}{\partial \xi} = W_\xi(\xi) = (\sqrt{1+\theta_i^2} - \theta_i)\xi^3 = \alpha_i \xi^3,$$

and

$$k(\xi) = W_\xi(\xi)/h(\xi) = 1/\alpha_i.$$

Then, a multi-estimator can be written as follows.

$$K_{P_i} \begin{cases} \dot{\xi}_i &= \theta_i \xi_i^3 - \frac{1}{\alpha_i}\xi_i^3 + \frac{y}{\alpha_i} + u \\ z &= y - \xi_i^3, \qquad\qquad i = 1, 2. \end{cases} \tag{4.9}$$

The next step is to design a controller for each of the two models. There are many ways to design such controller, but we consider use of a feedback linearisation controller for two reasons. Firstly, the linear system is easy to analyse and design. Especially when we want to achieve reference tracking and disturbance rejection by augmenting a small gain integrator, we need the "incremental DC gain" of the system to be uniformly bounded away from zero (see [70]). The linear system possesses this property. The second reason is related to paper [54], which uses a similar method to deal with the linear problem and achieves good results. Thus, if we can linearise our system by feedback, then some results can be achieved directly from paper [54].

Based on the above consideration, we design the stabilising controller as follows.

$$u_i = -\theta_i x^3 - 2x.$$

(where θ_1 or θ_2 is used according as the performance assessment concludes which one is the more likely.) Furthermore, we augment each controller with a small gain integrator. Of course this controller used on P_i would be satisfactory.

We implemented our design simulation by Matlab Simulink (see Figure 4.3).

The reference input in Figure 4.3 is a constant value 1.3. The input disturbance in Figure 4.3 is a constant value 0.2. From $t = 0$ to $t = 20_{sec}$, the value of the parameter θ in plant (4.7) is -0.2. At $t = 20_{sec}$, this value changes to 0.2, and the plant changes from open-loop stable to open-loop unstable. It should be mentioned that the plant is stable for the controller u_1 at $t < 20_{sec}$, and unstable with that controller afterwards. So, it is necessary to switch controller properly to ensure stability.

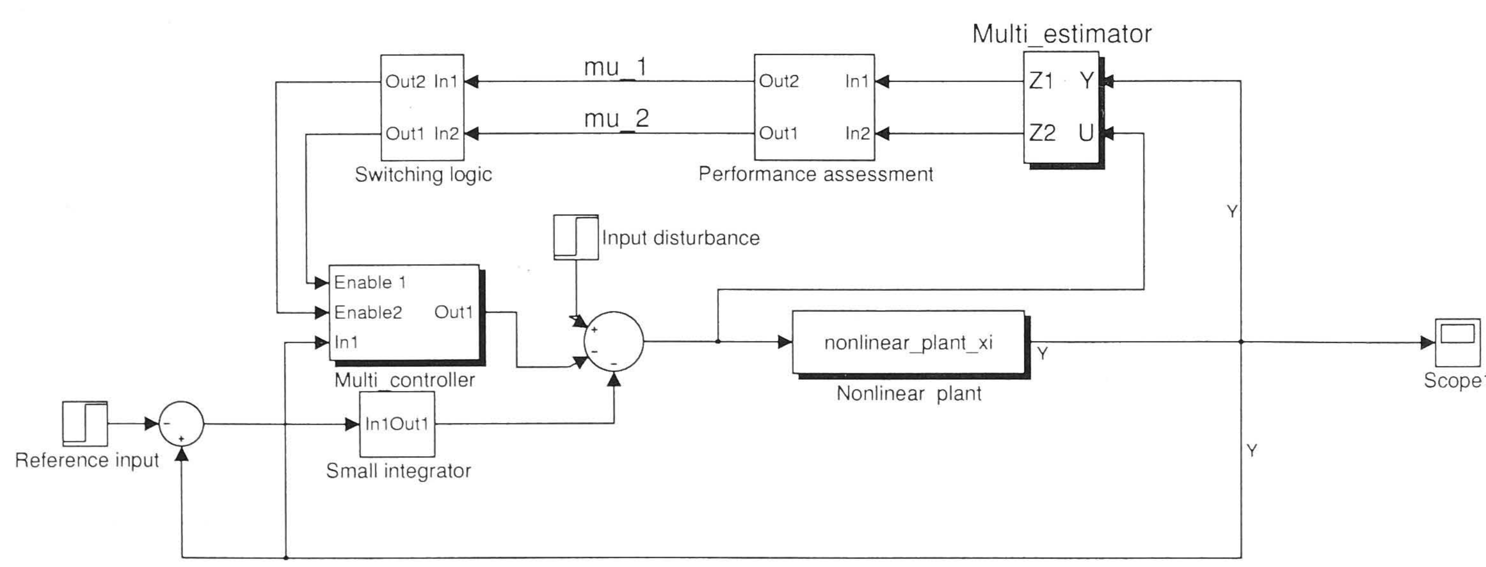

Figure 4.3: An example of MMAC for a nonlinear plant.

The simulation results are in Figure 4.4. The top figure in Figure 4.4 is the output y of the plant. The other two figures are the output of Performance assessment mu_1 and mu_2 respectively (See equation (4.6)). The integration of equation (4.6) is reseted to zero every 1 seconds.

From Figure 4.4, we can see that the multi-estimator accurately identified the variation of plant very fast. The whole control system can track a constant reference input under constant disturbance and switching model variant.

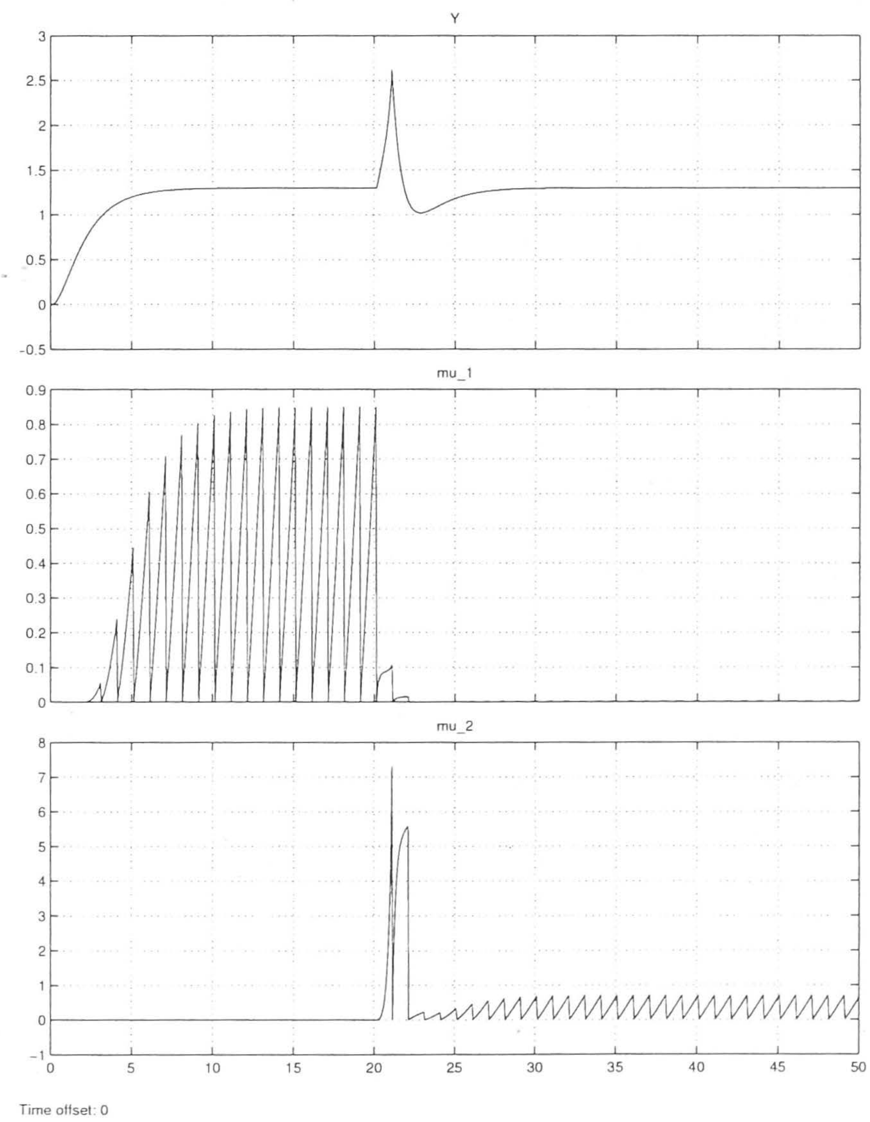

Figure 4.4: The output of the plant and multi-estimator.

Note 4.2 1. In this example, we have exhibited a multi-estimator design. An unaddressed issue is how the multi-controller and multi-estimator can be implemented more efficiently by using "state sharing" method (See [52]). (Actually, the two controllers in this example share the state of the small gain integrator.)

2. This simple example only uses two estimators and controllers. However, it is in principle easy to increase the number of estimators and controllers to deal with more difficult problems.

4.1.4 Conclusion

This section presents a modest extension of the Multiple Model Adaptive Control method in order to solve a robust constant disturbance rejection problem for nonlinear systems. We presented a method to construct a stable multi-estimator even for an open-loop unstable nonlinear system. Simulation results show that the designed controller can satisfactorily suppress a constant disturbance and track a constant reference under plant variation.

4.2 The minimal stably based multi-realisation of linear multi-variable systems

In this section, we present the efficient way of implementation of "multi-controller" and "multi-estimator" architectures. Multiple model adaptive control (MMAC) methods give rise to switching control systems. However, poor transient responses can result from controller switching [6]. One solution for poor transient responses is for controllers to share the state. For MMAC control systems, "multi-controller" and "multi-estimator" architectures which are realised as "state-shared" parameter-dependent feedback systems are also an efficient means of implementation. This section investigates the problem of the multi-realisation of a set of linear systems using parameter-dependent feedback to implement "state sharing" based on matrix fractional descriptions (MFDs). We present the results for the multi-realisation of a number of linear SISO systems, and highlight some fundamental issues such as the relationship between feedback multi-realisation and coprime factorisation. Then, necessary and sufficient conditions for the multi-realisation of a family of linear multi-variable systems are presented based on matrix fractional descriptions. Finally, the problem of the minimal multi-realisation (and "generic") of a set of linear systems is introduced and solved.

4.2.1 Introduction

Our original motivation for studying multi-realisation problems comes from multiple model adaptive control (MMAC) algorithms [5] [4] [29] [52] [54] [57]. As introduced in last section, MMAC is a model-based control strategy which incorporates a set of model/controller pairs rather than relying on a single model and controller to handle all possible operating conditions (see Figure 4.1). In more detail, MMAC algorithms assume that the unknown true plant either belongs to a given finite set of nominal plants, or is at least in some way close to one (or more) members of that set [5] [4] [52]. Each nominal plant corresponds to a controller that is presumed to give satisfactory performance in conjunction with both the nominal plant, and an associated uncertainty ball [5]. At any one instant of time, one controller alone is operational, namely the controller associated with that particular nominal plant which is judged by the performance assessment to be the closest to the true plant.

In order to make MMAC with its use of multi-estimator and multi-controller more efficient and practical, we try to find an efficient way of implementation for multi-controller and multi-estimator structure in this section. Just as one can consider a standard linear system realisation problem (given a transfer function, find a state-variable realisation), and a minimal realisation problem (ensure the state-variable realisation is of minimal degree), so for a finite collection of transfer functions can one consider a multi-realisation problem. The transfer functions here are those of the family of controllers or estimators. As argued, in for example [52], because at any instant of time only one of the constituent controllers is to be applied to the Plant, it is only necessary to generate one candidate control signal. Often this means significant simplification can be achieved if all control signals are generated by a single system. In other words, rather than implementing each of the controllers in the family as a separate dynamical system, one can often achieve the same results using a single controller with adjustable parameters (see Definition 4.6). Because the single controller state is, in effect, shared by the family of controllers, we call this implementation a state sharing multi-realisation using parameter dependent feedback. A well-known problem in switching control is the poor transient response that can arise due to controller switching. State sharing will ameliorate this kind of problem.

The implementation of a single linear time invariant (LTI) system has been extensively studied [7] [16] [17] [18] [23] [24] [37] [44] [48] [51] [61] [66] [81] [82] based on one of a state space description approach, matrix fraction description approach or a geometric approach. Morse [52] presented some results for the multi-realisation of several linear SISO systems in the context of examining MMAC for scalar plants. In this chapter, we investigate the multi-realisation of several linear multiple input multiple output (MIMO) systems; The results will be applicable to MMAC problems for MIMO plants. This is done as a first step towards a comprehensive theory of multi-controllers and multi-estimators for nonlinear systems.

For MMAC, it turns out that the left coprime factorisation representation of a transfer function matrix is especially important. One reason for this is that "bumpless" transfer between controllers will based on left coprime factorisation expressions (see Subsection 4.2.2). Another reason is that the construction of stable multi-estimators for even open-loop unstable plants can be based on stable **normalised** left coprime factorisations (or kernel representations).

In [5] and [53], multiple estimators for linear SISO systems were used to supervise the choice of models and controllers. The multi-estimators can also be realised efficiently by using the idea of state sharing.

In the next subsection, we introduce the concept of state sharing, and discuss the problem of the multi-realisation of a set of linear SISO systems. Some fundamental issues, including the relationship between feedback multi-realisation and coprime factorisation, are highlighted. Although some of these results are trivial for linear SISO systems, they suggest how to deal with the multi-variable case. In Subsection 4.2.3, we present the relationship between feedback multi-realisation and stable coprime factorisation for the multi-variable case. In Subsection 4.2.4, we present necessary and sufficient conditions for multi-realisation of multi-variable systems. Subsection 4.2.5 presents results for the minimal (generic) multi-realisation problem for any given set of linear systems with compatible input and output dimensions. Conclusions are given in Subsection 4.2.6.

4.2.2 Efficient multi-realisation for linear SISO systems

In this subsection, we highlight for SISO systems how multi-realisation problems can be solved by state sharing and feedback.

Suppose that it is desired to implement a finite number of SISO linear proper rational systems with transfer functions $\kappa_i(s) = \frac{n_i(s)}{d_i(s)} (i \in \mathcal{I})$, where $(n_i(s), d_i(s))$ are coprime polynomials. Assuming an upper bound n for the McMillan Degree of the $\kappa_i(s)$, we could realise each $\kappa_i(s)$ by using an n-dimensional state space description with adjustable parameters as in equation (4.10).

$$\Sigma_{C_i}\begin{cases}\dot{x}_C &= A_{q_i}x_C + b_{q_i}u \\ y &= c_{q_i}x_C + d_{q_i}u\end{cases} \tag{4.10}$$

We denote this n-dimensional SISO system as $\{A_{q_i}, b_{q_i}, c_{q_i}, d_{q_i}\}$.

Of course, it is possible to realise lower order transfer functions $\kappa_i(s)$ $(n_i < n)$ by allowing stable pole-zero cancellations, or equivalently by incorporating stable unobservable or uncontrollable modes in (4.10). Therefore, it is obvious that we can implement all the transfer functions $\kappa_i(s)$, if we choose the adjustable parameters $A_{q_i}, b_{q_i}, c_{q_i}, d_{q_i}$ properly. This implementation could be regarded as a state sharing multi-realisation because the state could be shared. If we were required to switch between different Σ_{C_i} at isolated instants of time, we could do this by switching parameters in A, b c and d, and therefore switching up to n^2+2n+1 parameters, while keeping x_c unchanged across the switching instant. However, we also need to consider efficient implementation. For efficiency, we aim for a small number of dependent parameters.

While a multi-realisation of $\kappa_i(s)$, $i \in \{1, \cdots, N\}$ is just a collection $\{A_{q_i}, b_{q_i}, c_{q_i}, d_{q_i}\}$, $i \in \{1, \cdots, N\}$ of state-variable realisation of $\kappa_i(s)$, our principal use of the term will effectively arise when there is only a "*minimal*" number of i-dependent parameters in each of the separate realisations.

In this chapter, we consider two multi-realisation forms which are dual, and we discuss the relationship between these two multi-realisations and coprime factorisation. We will later extend these two forms to the MIMO case.

> **Theorem 4.3** Consider a finite set of linear proper SISO systems described by transfer functions $\kappa_i(s) = \frac{n_i(s)}{d_i(s)}$, where an upper bound for the McMillan Degree of the $\kappa_i(s)$ $(i \in \mathcal{I})$ is n, and $(n_i(s), d_i(s))$ are coprime polynomials. Then there exists an n-dimensional controllable pair (A_0, b_0) with $Re\{\lambda_i(A_0)\} < 0$ such that $\{A_0 + b_0 k_{q_i}, b_0, c_{q_i}, d_{q_i}\}$ is a state space realisation of each transfer function $\kappa_i(s)$, with corresponding adjustable parameters $k_{q_i} \in \mathcal{R}^n, c_{q_i} \in \mathcal{R}^{1\times n}, d_{q_i} \in \mathcal{R}$, $i \in \{1, 2, \ldots, N\}$.

Proof

First, express each $\kappa_i(s)$ of order less than n as $\kappa_i(s)\frac{(s+a)^{\gamma_i}}{(s+a)^{\gamma_i}}$ for some $a > 0$, with the new denominator of degree n. Then, we describe each $\kappa_i(s)\frac{(s+a)^{\gamma_i}}{(s+a)^{\gamma_i}}$ by using the controller form realisation $\{A_{c_i}, b_0, c_{c_i}, d_{c_i}\}$ [37]; here, A_{c_i} is a companion matrix, with entries in the last row determining the characteristic polynomial and $b_0 = [0\,0 \cdots 1]^T$. Then observe that for any fixed companion matrix A_0, we have $A_{c_i} = A_0 + b_0 k_i$ for some k_i. Without loss of generality, $Re\{\lambda_i(A_0)\} < 0$ can be assumed. ∎

Likewise we have:

Theorem 4.4 (Dual of Theorem 4.3.) Consider a finite set of linear proper SISO systems described by transfer functions $\kappa_i(s) = \frac{n_i(s)}{d_i(s)}$, where an upper bound for the McMillan Degree of the $\kappa_i(s)$ $(i \in \mathcal{I})$ is n, and $(n_i(s), d_i(s))$ are coprime polynomials. Then there exists an n-dimensional observable pair (A_0, c_0) with $Re\{\lambda_i(A_0)\} < 0$ such that $\{A_0 + f_{q_i} c_0, b_{q_i}, c_0, d_{q_i}\}$ is a state space realisation of each transfer function $\kappa_i(s)$, with corresponding adjustable parameters $f_{q_i} \in \mathcal{R}^n, b_{q_i} \in \mathcal{R}^n, d_{q_i} \in \mathcal{R}$, $i \in \{1, 2, \ldots, N\}$.

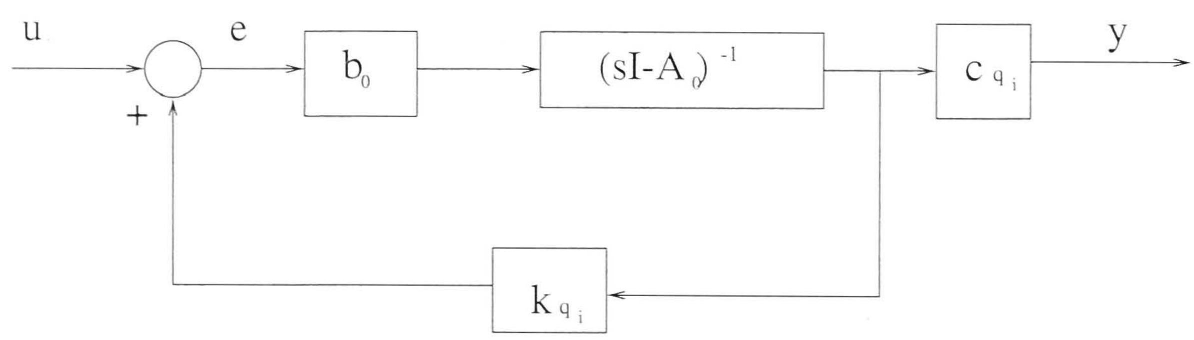

Figure 4.5: Right coprime factorisation of a SISO system

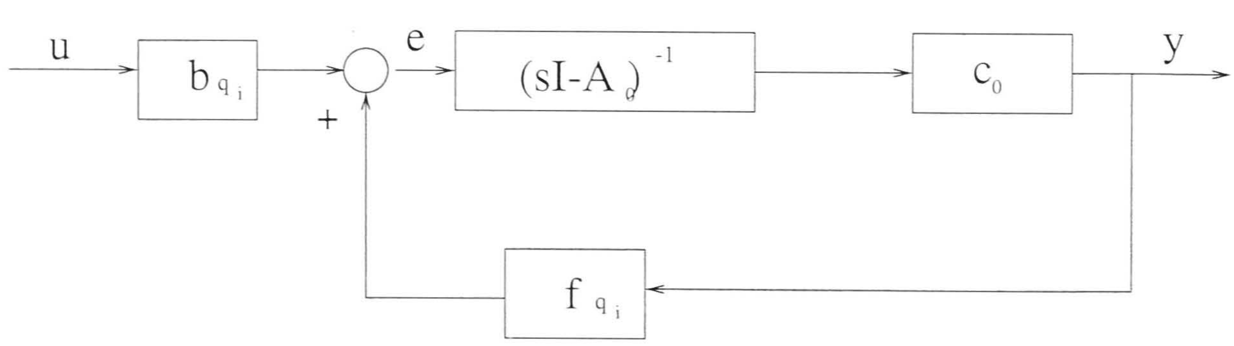

Figure 4.6: Left coprime factorisation of a SISO system

Note 4.5 The two multi-realisation forms of Theorems 4.3 and 4.4 can be implemented by feedback (See figures 4.5 and 4.6; which display the case where $d_{q_i} = 0$. The captions will be explained below.).

For the multi-realisation $\{A_0 + b_0 k_{q_i}, b_0, c_{q_i}, d_{q_i}\}$, we have

$$c_{q_i}[sI-(A_0+b_0k_{q_i})]^{-1}b_0=\frac{c_{q_i}(sI-A_0)^{-1}b_0}{1-c_{q_i}(sI-A_0)^{-1}k_{q_i}}.$$

This multi-realisation form can be implemented as in Figure 4.5. Although for linear SISO systems transfer function composition is commutative, so that there is virtually no difference between right and left coprime factorisation, this multi-realisation form corresponds to a right fractional description for MIMO systems. The multi-realisation solutions of Theorem 4.3 and 4.4 have simple interpretations (which will motivate the MIMO solution to follow) as coprime factorisations of stable proper transfer function. The fact that $c_{q_i}(sI-A_0)^{-1}b_0$ and $1+c_{q_i}(sI-A_0)^{-1}k_{q_i}$ are coprime over the Euclidean domain of stable proper transfer functions is a consequence of the construction of Theorem 4.3.

On the other hand, for the multi-realisation $\{A_0+f_{q_i}c_0, b_{q_i}, c_0, d_{q_i}\}$, we have

$$c_0(sI-(A_0+f_{q_i}c_0))^{-1}b_{q_i}=\frac{c_0(sI-A_0)^{-1}b_{q_i}}{1-c_0(sI-A_0)^{-1}f_{q_i}}$$

This multi-realisation can be implemented by feedback as depicted in Figure 4.6. This multi-realisation form corresponds to a stable left coprime factorisation expression for MIMO systems.

When the transfer functions in question correspond to multiple controllers which may be switched serially, the multi-realisation of Theorem 4.4 is preferable in the event that d_{q_i} is independent of i. This is because y remains continuous across switching instants, provided u is reasonably well behaved, e.g. is piecewise continuous, i.e. "*bumpless*" transfer [52] is achieved.

Now, we extend the concept of multi-realisation of a set of linear SISO systems to that of linear MIMO systems. We present a definition which defines the concept of a minimal stably based multi-realisation for multi-variable systems. The definition is based on the right coprime factorisation form $\{A_0+B_0K_{q_i}, B_0, C_{q_i}\}$ (see Theorem 4.3). By duality, it is possible to define the concept of a minimal stably based multi-realisation for the left coprime factorisation form.

Definition 4.6 Assume that there are given a number N of m-input p-output strictly proper real rational transfer function matrices P_i ($i \in \{1,2,\ldots,N\}$). Provided that there exist state variable realisations $\{A_0+$

$B_0K_{q_i}, B_0, C_{q_i}\}$ (with the pair (A_0, B_0) being controllable) that can realise all the systems P_i with adjustable parameters C_{q_i} and K_{q_i}, then we call $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ a multi-realisation of the set of systems P_i $(i \in \{1, 2, \ldots, N\})$. If all eigenvalues of A_0 are in the left half plane, we say that $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ is a stably based multi-realisation of the set of systems P_i $(i \in \{1, 2, \ldots, N\})$. Furthermore, if the dimension of A_0 is the smallest of all such stably based multi-realisations, then we call $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ a minimal stably based multi-realisation of the set of systems P_i $(i \in \{1, 2, \ldots, N\})$.

We now introduce a new operator $(\mathcal{D}_{hc}\{\cdot\})$ which will be used throughout our discussion.

Definition 4.7 Given a polynomial matrix $D(s)$, it is always possible to write

$$D(s) = D^{hc}S(s) + D_{lc}\Psi(s).$$

where, D^{hc} is the highest-degree-coefficient matrix of $D(s)$, D_{lc} is the lower-degree-coefficient matrix of $D(s)$, $S(s) \triangleq diag\{s^{k_1}, s^{k_2}, \cdots, s^{k_m}\}$, k_i is the degree of the i-th column of $D(s)$, and

$\Psi^T(s) \triangleq block\ diag\{[s^{k_1-1}, \cdots, s, 1], [s^{k_2-1}, \cdots, s, 1], \cdots, [s^{k_m-1}, \cdots, s, 1]\}.$

Define the operator $\mathcal{D}_{hc}(\cdot)$ as

$$\mathcal{D}_{hc}(D(s)) = D^{hc}S(s)$$

The degree of a square polynomial matrix in this chapter is defined as follows.

Definition 4.8 The degree of a square polynomial matrix $D(s)$ is defined as in [37]

$$deg(D(s)) = deg(det(D(s))).$$

4.2.3 State feedback and coprime factorisation for multi-variable systems

In Subsection 4.2.2, we showed that for scalar systems, the multi-realisation solutions of Theorem 4.3 and 4.4 have interpretations as coprime factorisations of stable proper transfer functions. In this subsection, we will investigate this topic for multi-variable systems.

In this chapter, we actually need to deal with the concept of right coprimeness of matrices over two different Euclidean domains (polynomial and proper (real) stable rational function S), i.e. polynomial matrices and proper (real) stable rational matrices (or matrices in $\mathcal{RH}_\infty$) [2] [37] [46] [78] [65] [68] [84].

Definition 4.9 Let $\mathcal{E}$ be a Euclidean domain (e.g. polynomial or proper (real) stable rational functions). Let M and N be matrices with entries in $\mathcal{E}$, and with the same number of columns. Then, M and N are termed right coprime if and only if one of the following equivalent conditions holds:

(a) If

$$\begin{bmatrix} M \\ N \end{bmatrix} = \begin{bmatrix} M_1 \\ N_1 \end{bmatrix} P \tag{4.11}$$

where M_1, N_1, P are matrices with entries in $\mathcal{E}$, then the entries of matrix P^{-1} are also in $\mathcal{E}$ (Equivalently, any greatest common right divisor is unimodular).

(b)There exist matrices X, Y with entries in $\mathcal{E}$ such that

$$XM + YN = I. \tag{4.12}$$

(c) i)For polynomial matrices, $[M(s)^T \, N(s)^T]$ has full column rank for all finite s.

ii)For matrices in $\mathcal{RH}_\infty$, $[M(s)^T \, N(s)^T]$ has full column rank for $s \in C_{+e}$, where C_{+e} denote the extended right half-plane, i.e. the closed right half plane together with the point at infinity.

The following Theorem is from [78].

Theorem 4.10 Consider the system described by equation (4.13) following.

$$P(s) = C(sI - A)^{-1}B \tag{4.13}$$

where A, B, C are constant matrices of compatible dimensions, and the pairs (A, B) and (A, C) are stabilisable and detectable, respectively. Given constant matrices K and F such that the matrices $A_0 = A + BK$, $\tilde{A}_0 = A + FC$ have all open left plane eigenvalues, then $P = N_g D_g^{-1} = \tilde{D}_g^{-1}\tilde{N}_g$ where

$$\begin{cases} \tilde{N}_g = C(sI - \tilde{A}_0)^{-1}B \\ \tilde{D}_g = I + C(sI - \tilde{A}_0)^{-1}F. \end{cases} \tag{4.14}$$

$$\begin{cases} N_g = C(sI - A_0)^{-1}B \\ D_g = I + K(sI - A_0)^{-1}B. \end{cases} \tag{4.15}$$

and the two fractional descriptions are coprime over $\mathcal{RH}_\infty$.

Proof See [78] ∎

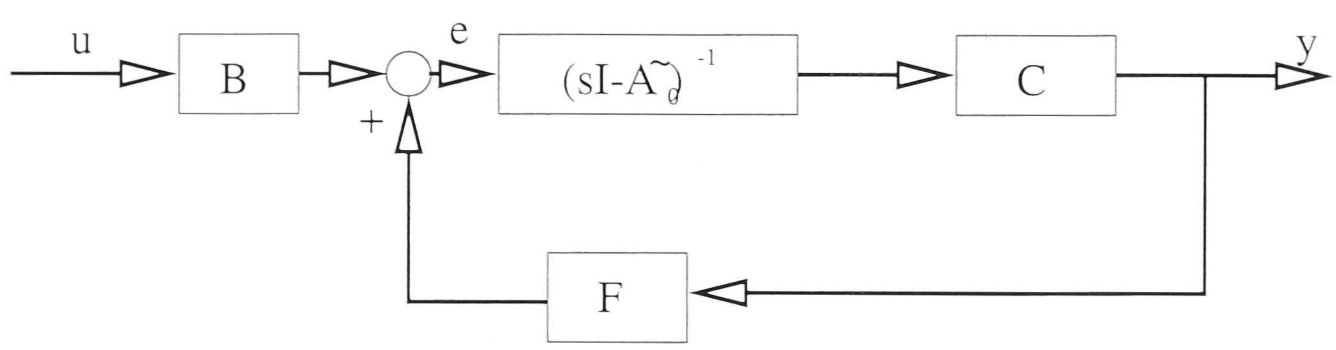

Figure 4.7: Left coprime factorisation for MIMO systems

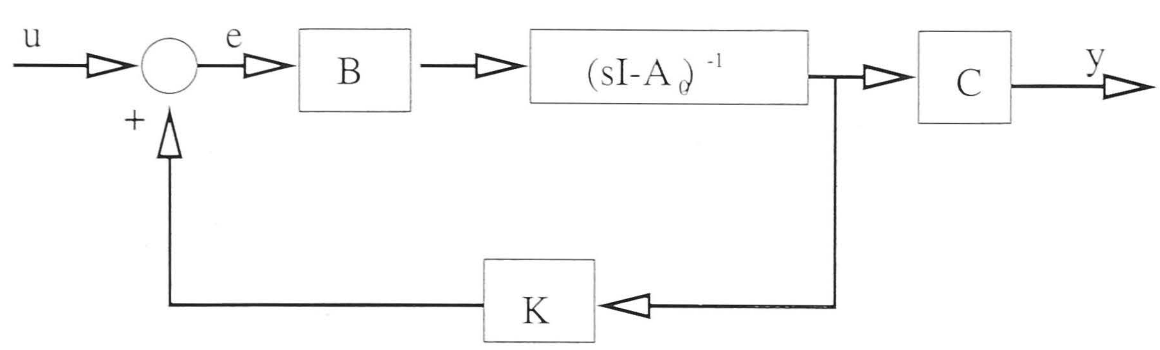

Figure 4.8: Right coprime factorisation for MIMO systems

Considering the form $A_0 = A + BK$, $\tilde{A}_0 = A + FC$, we can see that Figure 4.7 and Figure 4.8 correspond to left coprime factorisation and right coprime factorisation respectively.

Note 4.11 1)Theorem 4.10 actually describes the relationship between feedback realisations and coprime factorisations.
2)It should be noted that the right (left) coprime factorisation property means that the subsystem transfer function matrices (fractional matrices in $\mathcal{RH}_\infty$) N_g and D_g ($\tilde{N}_g$ and $\tilde{D}_g$) are right (left) coprime. However, the

state variable realisation of the subsystems N_g and D_g ($\tilde{N}_g$ and $\tilde{D}_g$) described by equation (4.14) and (4.15) are not necessarily minimal because we have assumed merely that the pairs (A, B) and (A, C) are stabilisable and detectable.

4.2.4 Conditions for the existence of multi-variable system multi-realisations

In this subsection, we investigate the problem of achieving a stably based multi-realisation of a set of MIMO systems (see Definition 4.6), and give sufficient and necessary conditions for multi-realisation of MIMO systems. The minimal stably based multi-realisation problem will be considered in Subsection 4.2.5.

The invariant description of linear multi-variable systems

We review the invariant description of linear multi-variable systems given by Popov in [59] in this part. Based on this invariant description we can directly achieve sufficient and necessary conditions for the (efficient) multi-realisation of multi-variable systems.

Let $H(s) = C(sI-A)^{-1}B$ be a strictly proper MIMO real rational transfer function where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$, the pair (A, B) is controllable, and B has full column rank.

Consider the ordered set of vectors

$$b_1, \cdots, b_m, Ab_1, \cdots, Ab_m, A^2b_1, \cdots, A^2b_m, \cdots, \tag{4.16}$$

where the b_i are the columns of matrix B, written as $B = [b_1\, b_2 \,\cdots\, b_m]$.
We shall say that a vector A^kb_j from (4.16) is an antecedent of another vector A^pb_q from (4.16) if and only if A^kb_j is situated before A^pb_q in (4.16).

Definition 4.12 For every integer $i \in \{1, \cdots, m\}$, define the i-th Kronecker invariant (Popov integer parameters), d_i, as the smallest positive integer such that the vector $A^{d_i}b_i$ is a linear combination of its antecedents. The integers $d_1, \cdots, d_m$ (with their ordering) are termed the controllability indices of the pair (A, B). Furthermore, define

$$\sigma_k = \sum_{i=1}^{k} d_i \tag{4.17}$$

for $k \in \{1, 2, \ldots, m\}$.

Lemma 4.13 The Kronecker invariants (controllability indices) from Definition 4.12 satisfy the relation

$$d_1 + d_2 + \cdots + d_m = n. \tag{4.18}$$

where n is dimension of the minimal state space realisation.

Proof See Proposition 3 of [59]. ∎

Lemma 4.14 There exists exactly one set of ordered numbers (Popov parameters) $\alpha_{ijk} \in \mathcal{R}$, defined for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, i-1$, $k = 0, 1, \cdots, min(d_i, d_j - 1)$ and for $i = 1, 2, \cdots, m$, $j = i, i+1, \cdots, m$, $k = 0, 1, \cdots, min(d_i, d_j) - 1$ such that, for every $i = 1, 2, \cdots, m$, one has the dependency relations

$$A^{d_i} b_i = \sum_{j=1}^{i-1} \sum_{k=0}^{min(d_i,d_j-1)} \alpha_{ijk} A^k b_j + \sum_{j=i}^{m} \sum_{k=0}^{min(d_i,d_j)-1} \alpha_{ijk} A^k b_j. \tag{4.19}$$

Proof See corollary 1 of [59]. ∎

Theorem 4.15 Suppose that $(A, B) \in \mathcal{R}^{n \times (n+m)}$ is a controllable pair, and B has full column rank. Let $\beta_{ijk} \in \mathcal{R}$ be arbitrary real numbers, defined for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, i-1$, $k = 0, 1, \cdots, min(d_i, d_j) - 1$. Then there exists a matrix $K \in \mathcal{R}^{m \times n}$ such that the Kronecker invariants $\tilde{d}_i$ and Popov parameters $\tilde{\alpha}_{ijk}$ of the pair of matrices $(A + BK, B)$ are given by the relations

$$\tilde{d}_i = d_i, i = 1, 2, \cdots, m, \tag{4.20}$$

$$\tilde{\alpha}_{ijk} = \beta_{ijk}, for\, i, j = 1, 2, \cdots, m,\ k = 0, 1, \cdots, min(d_i, d_j) - 1, \tag{4.21}$$

$$\tilde{\alpha}_{ijk} = \alpha_{ijd_i}, for\ k = d_i, i = 1, 2, \cdots, m, \tag{4.22}$$

and for every integer $j \in \{1, \cdots, i-1\}$ for which $d_j > d_i$.

Proof See Theorem 3 of [59]. ∎

Note 4.16 1) Theorem 4.15 shows that the invariants from (4.20) and (4.22) constitute a complete system of independent invariants for the pair (A, B) with respect to the transformations $\tilde{A} = T(A+BK)T^{-1}$, $\tilde{B} = TB$. 2) It also can be seen that a complete system of independent invariants with respect to the transformations $\tilde{A} = T(A+BK)T^{-1}$, $\tilde{B} = TBG$ (where $G \in \mathcal{R}^{m \times m}$ is an arbitrary nonsingular matrix) is given by the **unordered** set of d_i. This limited case gives the Brunovsky classes from [15].

Conditions for multi-realisation of multi-variable systems based on MFDs

In order to achieve the conditions for the multi-realisation of multi-variable systems, we will work a multi-variable canonical form in a polynomial matrix fractional description of a transfer function matrix.

We shall say that a square polynomial matrix $D_E(s)$ is in polynomial-echelon or Popov form if it has the following characteristics [37].

1. It is column reduced, with its column degrees arranged in ascending order, say

$$k_1 \leq k_2 \leq \ldots \leq k_m.$$

2. For column j, $1 \leq j \leq m$, there is a so-called **pivot index** p_j such that

a. $d_{p_j j}(s)$ has degree k_j. Here, $d_{ij}(s)$ is the element in the i-th row and j-th column of polynomial matrix $D(s)$.

b. $d_{p_j j}(s)$ is monic.

c. $d_{p_j j}(s)$ is the last (or lowest) entry of degree k_j in the j-th column; i.e., $deg d_{ij}(s) < k_j$ if $i > p_j$.

d. If $k_i = k_j$ and $i < j$, then $p_i < p_j$; i.e., the pivot indices are arranged to be increasing.

e. $d_{p_j i}(s)$ has degree less than k_j if $i \neq j$.

Now, let $H(s) = N(s)D^{-1}(s)$ be a proper MIMO real rational transfer function, where $N(s)$ and $D(s)$ are real polynomial matrices; we do *not* require that $N(s)$ and $D(s)$ are, as polynomial matrices, coprime. Then there exists a real polynomial matrix $D_E(s)$, in polynomial echelon form, as defined above, such that

$$H(s) = N_E(s)D_E^{-1}(s), \tag{4.23}$$

$$\begin{bmatrix} N_E(s) \\ D_E(s) \end{bmatrix} = \begin{bmatrix} N(s) \\ D(s) \end{bmatrix} V(s) \tag{4.24}$$

with $V(s)$ a unimodular real polynomial matrix (thus $V(s)$ has a nonzero constant determinant). Further, $D_E(s)$ is unique, in the sense that $D(s)$ and $D(s)\bar{V}(s)$ have the same Popov form for any unimodular $\bar{V}(s)$.

The relationship between invariant Popov parameters α_{ijk} and the Popov form $D_E(s)$ can now be stated:

Theorem 4.17 For a Popov form polynomial matrix $D_E(s)$, if we denote

$$D_E(s) = [d_{ij}(s)] = [\sum_{l=0}^{k_j} d_{ijl} s^l] \tag{4.25}$$

then

$$d_{ijl} = \begin{cases} 1, & if\, l = k_j\, and\, i = p_j, \\ -\alpha_{p_j i l} & \{l < k_j\}\, or \{l = k_j\, and\, i < p_j\}. \end{cases} \tag{4.26}$$

and d_{ijl} otherwise is zero. Here, the $\{\alpha_{ijk}\}$ are the Popov parameters describing the dependency relations (4.19) of any controllable state variable realisation of $D_E^{-1}(s)$.

Proof See equation (17) and associated statements in Page 482 of [37]. ∎

The following theorem will present the relationship between the column degrees of a Popov polynomial matrix $D_E(s)$ and the controllability indices of a controllable pair (A, B) of a minimal state variable realisation of $D_E^{-1}(s)$.

Theorem 4.18 Consider a strictly proper multi-variable system $H(s)$ described by a polynomial right MFD, i.e. $H(s) = N_E(s)D_E^{-1}(s)$ where $D_E(s)$ is a Popov polynomial matrix. Let k_i denote the i_{th} column degree of the Popov polynomial matrix $D_E(s)$, p_j denote the pivot index of the j_{th} column of the Popov polynomial matrix $D_E(s)$, and d_i denote the i_{th} controllability index of a controllable pair (A, B) of a minimal state variable realisation of $D_E^{-1}(s)$. Then

i)$k_i = d_{p_i}$.

ii)The real matrix D_E^{hc}, the highest-degree-coefficient matrix of the polynomial matrix $D_E(s)$, is the identity matrix. i.e. $D_E^{hc} = I$, if and only if the i_{th} column pivot index of the polynomial matrix $D_E(s)$ is equal to i (That is equivalent, according to i) of this theorem, to the condition $d_1 \leq d_2 \leq \cdots \leq d_m$).

Proof i) Through post multiplication by a real matrix R, the columns of the Popov polynomial matrix $D_E(s)$ can be reordered according to the pivot index of each column. More specifically, the i_{th} pivot index of the reordered polynomial matrix is equal to i. If we denote $\tilde{D}(s) = D_E(s)R$, and $\tilde{k}_i$ as the i_{th} column degree of the reordered polynomial matrix $\tilde{D}(s)$, then,

$$k_i = \tilde{k}_{p_i}. \tag{4.27}$$

It is easy to see that $\tilde{D}^{hc}$, the highest-degree-coefficient of polynomial matrix $\tilde{D}(s)$ is an upper triangular matrix.

Then, we realise the right MFDs $H(s) = \tilde{N}(s)\tilde{D}_E^{-1}(s)$ by $\{A_c, B_c, C_c\}$, which is a controller form realisation by using the method in Page403-407 of [37]. Since we know that $\tilde{D}^{hc}$ is an upper triangular matrix, we can check that the controllability indices of the controllable pair (A_c, B_c) are

$$d_i = \tilde{k}_i \tag{4.28}$$

according to equation (8)-(10) in Page406-407 of [37] and the associated discussion. We conclude $k_i = d_{p_i}$ based on equations (4.27) and (4.28).

ii)The necessity is obvious. We prove the sufficency here. If for each i the i_{th} column pivot index of the polynomial matrix $D_E(s)$ is equal to i, then according to 2.c of the description of a Popov form polynomial matrix, we conclude that D_E^{hc} is an upper triangular matrix. Furthermore, according to 2.b and 2.e (all entries in a row containing a pivot element have degree lower than that of the pivot element.) of the description of a Popov form polynomial matrix, we conclude that $D_E^{hc} = I$. ∎

Corresponding to the scalar case, there are also two dual multi-realisation forms for multi-variable systems: $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ (corresponding to a right coprime factorisation) and $\{A_0 + F_{q_i}C_0, B_{q_i}, C_0\}$ (corresponding to a left coprime factorisation).

The realisation form $\{A_0 + F_{q_i}C_0, B_{q_i}, C_0\}$ is more important for implementation because it ensures "*bumpless*" switching, as explained for the scalar case. However, it is more convenient to discuss the realisation form $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ because for this multi-realisation form we can directly apply insights derived from the invariant description introduced earlier.

Now, we present sufficient and necessary conditions for the existence of a multi-realisation of given MIMO systems based on polynomial matrix fractional descriptions for the realisation form $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$.

Theorem 4.19 For a set of m-input p-output strictly proper systems $H_i(s)$ ($i \in \{1, 2, \cdots, N\}$), there exists a controllable pair (A_0, B_0) ($dim\{A_0\} = n$), and appropriately dimensioned real matrices C_{q_i} and K_{q_i} (for $i \in \{1, 2, \ldots, N\}$) such that A_0 is stable, and $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ is a controllable realization of system $H_i(s)$, (for $i \in \{1, 2, \ldots, N\}$), if and only if, there exists a right polynomial MFDs for each system $H_i(s)$ described by $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ (where $D_{Ei}(s)$ is a Popov polynomial matrix with same degree n, i.e. $deg\{D_{Ei}(s)\} = n, \forall i \in \{1, 2, \cdots, N\}$) such that

i)$k_{il} = k_{jl}$ for $i, j \in \{1, 2, \ldots, N\}$ and $l \in \{1, 2, \ldots, m\}$, where k_{ij} is the j_{th} column degree of the matrix $D_{Ei}(s)$.

ii)All D_{Ei}^{hc} (for $i \in \{1, 2, \ldots, N\}$), the highest-degree-coefficient matrix of $D_{Ei}(s)$ (the Popov form of $D_i(s)$) are identical for each i.

Proof Assume first the existence of the controllable state variable multi-realisation for $H_i(s)$ $(i \in \{1, 2, \cdots, N\})$. It is standard that there exists a column reduced polynomial matrix $D_i(s)$ with $detD_i(s) = det(sI - A_0 - B_0K_{q_i})$ and with column degrees corresponding, apart possibly for ordering, with the controllability indices d_i of (A_0, B_0) which are the same as those of $\{A_0 + B_0K_{q_i}, B_0\}$ (see [37] or [81]). Further, $D_i(s)$ is such that for any constant matrix $F \in \mathcal{R}^{p\times n}$, there exists an associated $N_F(s)_{p\times m}$ such that

$$F(sI - A_0 - B_0K_{q_i})^{-1}B_0 = N_F(s)D_i^{-1}(s).$$

Conversely for any polynomail matrix $N_F(s)$ such that $N_F(s)D_i^{-1}(s)$ is strictly proper, there exists a real matrix F satisfing this equation.

Clearly, there exists a polynomial $N_i(s)$ such that $H_i(s) = N_i(s)D_i^{-1}(s)$. Without loss of generality, we can realise $D_i(s)$ by its Popov form $D_{Ei}(s)$, so that $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$. Futher, the column degrees of each $D_{Ei}(s)$ are the controllability indices of (A_0, B_0) (apart from ordering). In fact, with k_{ij} the column degree of the j_{th} column of $D_{Ei}(s)$, there holds $k_{ij} = d_{ip_j}$, by Theorem 4.18, where p_j is the pivot index for column j.

By Theorem 4.15, the Popov parameters α_{ljd_l} of $\{A_0 + B_0K_{q_i}, B_0\}$ and $\{A_0, B_0\}$ are the same for $j \in \{1, \cdots, l-1\}$ (and $d_j > d_l$). Equivalently, the parameters $\alpha_{p_ljd_{p_l}}$ are the same for $j < p_l$ (and $d_{p_j} > d_{p_l}$).

Now in $D_{Ei}(s)$, the j_{th} column for all i has maximum degree k_j by equation (4.26). Recalling (4.25), we see that the associated column of D_{Ei}^{hc} is (for each of the $D_{Ei}(s)$)

$$\begin{aligned} & [d_{1jk_j}\ d_{2jk_j}\ \cdots\ d_{p_jjk_j}\ 0\ \cdots\ 0]^T \\ = & \ [-\alpha_{p_j1d_{p_j}}\ -\alpha_{p_j2d_{p_j}}\ \cdots\ -\alpha_{p_j,p_j-1,d_{p_j}}\ 1\ 0\ \cdots\ 0]^T \end{aligned} \tag{4.29}$$

which is the same for all $D_{Ei}(s)$. This proves claim ii).

Conversely, suppose there exist right polynomial MFDs $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ where D_{Ei} is a Popov polynomial matrix of degree n for all i, and the other conditions of the theorem statement hold. Let (A_i, B_i) be a completely controllable pair in a state variable realisation of $D_{Ei}^{-1}(s)$. Theorem 4.17 and the hypothesis imply that the (A_i, B_i) pairs have the same controllability indices and the invariants α_{ljd_l} for $j < l$ are the same. Accordingly, by Theorem 4.15 (see also Note 4.16), linking any two pairs (A_i, B_i) and (A_j, B_j) there exists a nonsingular matrix T_{ij} and K_{ij} such that

$$A_i = T_{ij}(A_j + B_j K_j)T_{ij}^{-1}, and B_i = T_{ij}B_j.$$

Equivalently, there exists (A_0, B_0), K_{q_i} and C_{q_i} as in the Theorem statement. ∎

By observing the duality between the two multi-realisations $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ and $\{A_0 + F_{q_i}C_0, B_{q_i}, C_0\}$, it is easy to achieve sufficient and necessary conditions for the multi-realisation form $\{A_0 + F_{q_i}C_0, B_{q_i}, C_0\}$ from Theorem 4.19. The only difficulty is to carefully set up the corresponding relationship between column-related properties and row-related properties. Throughout this chapter, we will only discuss the multi-realisation form $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$. Corresponding results for the multi-realisation form $\{A_0 + F_{q_i}C_0, B_{q_i}, C_0\}$ can be easily achieved by using the duality relationship.

4.2.5 Minimal stably based multi-realisation for multi-variable systems

In Theorem 4.19, we presented necessary and sufficient conditions for an efficient multi-realisation of a set of multi-variable systems. However, the conditions in Theorem 4.19 may not be satisfied if we restrain the dimension of the state variable multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$. Another way to consider this multi-realisation problem is to allow available an input transformation (as stated in point 2 of Note 4.16), that is, to consider the state variable multi-realisation of the form $\{A_0 + B_0K_{q_i}, B_0G_{qi}, C_{q_i}\}$ (where $G_{qi} \in \mathcal{R}^{m\times m}$ is an adjustable nonsingular matrix). However, neither the the state variable multi-realisation $\{A_0 + B_0K_{q_i}, B_0G_{qi}, C_{q_i}\}$ nor its dual form $\{A_0 + F_{q_i}C_0, B_{q_i}, Q_{q_i}C_0\}$ (where $Q_{qi} \in \mathcal{R}^{p\times p}$ is an adjustable nonsingular matrix) will ensure "*bumpless*" switching.

In this part, we consider the minimal stably based multi-realisation problem (see Definition 4.6). For this multi-realisation problem, the dimension of the state variable multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ is not fixed but minimised. We will discuss two kinds of minimal stably based multi-realisations. The dual state variable multi-realisation $\{A_0 + F_{q_i}C_0, B_{q_i}, C_0\}$ will ensure "*bumpless*" switching as mentioned in Subsection 4.2.4.

Problem description

In Subsection 4.2.2, Definition 4.6 describes the concept of minimal stably based multi-realisation for multi-variable systems. There is another concept of minimal stably based multi-realisation which we call a minimal stably based "*generic*" multi-realisation, with distinction to be discussed later.

In order to simplify our discussion, we present a problem that is equivalent to the minimal stably based multi-realisation problem. We call it the "minimal common hc-(highest column degree) multiplier problem" for a set of polynomial matrices.

Problem 4.20 Given a finite set of square $(m \times m)$ column-reduced polynomial matrices $D_i(s)$, find nonsingular stable polynomial matrices $X_i(s)$ (That is, the zeros of $det(X_i(s))$ lie in the left half plane $Re(s) < 0$) such that there exists a column reduced polynomial matrix $D_{min}(s)$ with the property that

$$\mathcal{D}_{hc}(D_i(s)X_i(s)) = D_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\}, \tag{4.30}$$

and $D_{min}(s)$ has the lowest possible degree.

Although the minimal common hc- (highest column degree) multiplier problem is actually equivalent to the minimal stably based multi-realisation problem, here we are only particularly interested in whether it is possible to construct the minimal stably based multi-realisation from the solution of the minimal common hc- (highest column degree) multiplier problem. The proof of the following theorem shows by construction that this is possible.

Theorem 4.21 Consider a set of m-input p-output strictly proper systems $H_i(s)$ $(i \in \{1, 2, \cdots, N\})$ described by right polynomial MFDs, i.e. $H_i(s) = N_i(s)D_i^{-1}(s)$, and $(N_i(s), D_i(s))$ are right coprime polynomial matrices. If for the set of polynomial matrices $D_i(s)$, one can find a minimal common hc-multiplier (as stated in Problem 4.20) $D_{min}(s)$, i.e. the column reduced polynomial matrix $D_{min}(s)$ satisfies equation (4.30) with the lowest possible degree, then, a minimal stably based multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ with $dim\{A_0\} = deg\{D_{min}\}$ for the set of systems $H_i(s)$ can be constructed.

Proof We use two steps to prove this theorem. The first step is to construct a stably based multi-realisation with $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ with $dim\{A_0\} = deg\{D_{min}\}$. The second step is to show this multi-realisation is minimal. Firstly, for each system $H_i(s) = N_i(s)D_i^{-1}(s)$, it is always true that for nonsingular $X_i(s)$,

$$H_i(s) = N_i(s)X_i(s)(D_i(s)X_i(s))^{-1} = \tilde{N}_i(s)\tilde{D}_i^{-1}(s). \tag{4.31}$$

We take $X_i(s)$ as those that solve Problem 4.20 and define $\tilde{N}_i(s) = N_i(s)X_i(s)$ and $\tilde{D}_i(s) = D_i(s)X_i(s)$ respectively. We consider the multi-realisation of systems $H_i(s) = \tilde{N}_i(s)\tilde{D}_i^{-1}(s)$.

We can construct a controller form realisation $\{\bar{A}_0, B_0, C_{q_i}\}$ (with the pair $(\bar{A}_0, B_0)$ controllable) for each system

$$\bar{H}_i(s) = N_i(s)D_{min}^{-1}(s).$$

Furthermore, even if $\bar{A}_0$ is not stable, because $\{\bar{A}_0, B_0\}$ is controllable we can easily find a stable A_0 and a adjustable feedback gain matrix K_0 such that $\bar{A}_0 = A_0 + B_0K_0$. We may thus construct a stably based multi-realisation $\{A_0 + B_0K_0, B_0, C_{q_i}\}$ with $dim\{A_0\} = deg\{D_{min}\}$ for the set of systems $\bar{H}_i(s)$.

Recall that the lower-degree coefficients of $D_{min}(s)$ can be completely changed by state feedback (see item (6) in Page 508 of [37] and equation (4.21) in Theorem 4.15), and $\mathcal{D}_{hc}\{D_i(s)X_i(s)\} = D_{min}(s)$, i.e. the highest-degree-coefficient matrix for $\tilde{D}_i(s)$ is equal to that for $D_{min}(s)$ for all $i \in \{1, 2, \cdots, N\}$. According to items (1)-(4) on Pages 507-508 of [37], we can see that

$$H_i(s) = N_i(s)D_i^{-1}(s)$$

is obtainable by state feedback. That is, there exists a feedback gain matrix K'_{qi} such that

$$\{A_0 + B_0(K_0 + K'_{q_i}), B_0, C_{q_i}\} = \{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}, with\, A_0\, stable,$$

is a state variable realisation of each system $H_i(s)$.

Next, we prove by contradiction that this stably based multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ with $dim A_0 = deg\{D_{min}\}$ for the systems $H_i(s)$ is minimal.

The proof is by contradiction. **Assume** there exists another stably based multi-realisation $\{\tilde{A}_0+\tilde{B}_0\tilde{K}_{q_i}, \tilde{B}_0, \tilde{C}_{q_i}\}$ for the systems $H_i(s)$, with $dim\{\tilde{A}_0\} < deg\{D_{min}(s)\}$.

Then, considering the necessary conditions for multi-realisation of a set of systems (see Theorem 4.19), we conclude that there exist right polynomial MFDs for each system $H_i(s)$ described by $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s)$ (where $D_{Ei}(s)$ is a Popov polynomial matrix with $deg\{D_{Ei}(s)\} = dim\{\tilde{A}_0\}, \forall i \in \{1, 2, \cdots, N\}$) such that
(a)$k_{il} = k_{jl}$ for $i, j \in \{1, 2, \ldots, N\}$ and $l \in \{1, 2, \ldots, m\}$, where k_{ij} is the j_{th} column degree of matrix $D_{Ei}(s)$.
(b)All D_{Ei}^{hc} (for $i \in \{1, 2, \ldots, N\}$), the highest-degree-coefficient matrix of $D_{Ei}(s)$ (the Popov form of $D_i(s)$), are identical.

Further consider that $H_i(s) = N_{Ei}(s)D_{Ei}^{-1}(s) = N_i(s)D_i^{-1}(s)$ and $(N_i(s), D_i(s))$ are right coprime polynomial matrices. We conclude that there exist a matix $\tilde{X}_i(s)$ such that

$$\begin{aligned} N_{Ei}(s) &= N_i(s)\tilde{X}_i(s), \\ D_{Ei}(s) &= D_i(s)\tilde{X}_i(s). \end{aligned} \tag{4.32}$$

Then, from (a), (b) and equation (4.32), we can see that for all $i \in \{1, 2, \cdots, N\}$

$$\mathcal{D}_{hc}\{D_i(s)\tilde{X}_i(s)\} = \mathcal{D}_{hc}\{D_{Ei}(s)\} = \tilde{D}_{minE}(s).$$

with $deg\{\tilde{D}_{minE}(s)\} = dim\{\tilde{A}_0\} < deg\{D_{min}(s)\}$. This **contradicts** the assumption that the matrix $D_{min}(s)$ is a minimal common hc-multiplier. Hence, the stably based multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ with $dim\{A_0\} = deg\{D_{min}\}$ for systems $H_i(s)$ is minimal. ∎

A solution for the minimal common hc-multiplier problem (Problem 4.20) will be given in Subsection 4.2.5.

Next, we will introduce an alternative and related concept of a minimal stably based multi-realisation which we call a minimal stably based "*generic*" multi-realisation.

The necessary and sufficient conditions for multi-realisation of multi-variable systems defined by transfer function matrices $H_i(s)$, $i = 1, \cdots, N$, (presented in Theorem 4.19) are that each $H_i(s)$ can be written as a right polynomial MFDs $N_{Ei}(s)D_{Ei}^{-1}(s)$ where D_{Ei} is a Popov form polynomial matrix, and all D_{Ei}^{hc}, the highest-degree-coefficient matrix of $D_{Ei}(s)$, are identical, and the column degrees of the $D_{Ei}(s)$ (including ordering) are the same. Equivalently, the controllability indices of a controllable pair (A_i, B_i) of a minimal state variable realisation of each $D_{Ei}^{-1}(s)$ are the same (with ordering) for all i.

Theorem 4.17 and Theorem 4.18 provide the connections between the matrix D_{Ei} and Popov parameters(defined in Definition 4.12 and Definition 4.14) of (A_i, B_i). By using Theorems 4.17, 4.18, and 4.19, we can see that the minimal stably based multi-realisation form $D_{min}(s)$ will be determined not only by the Popov integer parameters (consider that the controllability indices are defined as Popov integer parameters) but also by some of the Popov real parameters $\{\alpha^i_{ljd_l}\}$ for $l \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, j-1\}$ and $i \in \{1, 2, \cdots, N\}$ (see the relationship between the matrix D^{hc}_{Ei} and Popov real parameters $\{\alpha^i_{ljd_l}\}$ from equation (4.29)), whose values are invariant under state feedback.

For a minimal stably based "*generic*" multi-realisation, we aim to achieve a multi-realisation form $\bar{D}_{min}(s)$, which is independent of all the Popov real parameters of each multi-variable systems defined by transfer function matrices $H_i(s) = N_{Ei}(s)D^{-1}_{Ei}(s)$, $i = 1, \cdots, N$. The Popov real parameters are determined by physical parameters, which are prone to vary in application. Popov integer parameters (see Definition 4.12) however are related to the number of integrators and the structure in a physical realisation of each of the transfer function matrix $H_i(s) = N_{Ei}(s)D^{-1}_{Ei}(s)$, $i = 1, \cdots, N$. Popov integer parameters, defined upon the structure of the physical realisation, not the particular real value of a physical parameter, are relatively robust to modelling errors that arise due to parameters drift. So, the minimal stably based "*generic*" multi-realisation has significant relevance in practical application.

Theorem 4.18 states that the property that the controllability indices of a minimal state variable description for each $D^{-1}_i(s)$ increase column-wise is equivalent to the property that $D^{hc}_{Ei} = I$. This implies that if a controllable pair (A_i, B_i) of a minimal state variable realisation of each $D^{-1}_i(s)$ has the same increasingly ordered controllability indices (Popov integer parameters), then the feedback invariant Popov real parameters $\{\alpha^i_{ljd_l}\}$ (for the controllable pair (A_i, B_i)) will be identical for each i. Thus the two conditions for the multi-realisation of multi-variable systems (presented in Theorem 4.19) will be simultaneously satisfied. In other words, given that the controllability indices (Popov integer parameters) are increasingly ordered for each minimal realisation of $D^{-1}_i(s)$ ($i \in \{1, 2, \ldots, N\}$) (equivalent to $D^{hc}_{Ei} = I$), then the minimal multi-realisation of the set of transfer functions $H_i(s)$ is independent of all the Popov real parameters $\{\alpha^i_{ljk}\}$.

Based on this observation, we introduce another problem, "a generic minimal common hc-multiplier problem". It can be regarded as a minimal common hc-multiplier problem with an extra constraint.

Problem 4.22 Given a set of square $(m \times m)$ column-reduced polynomial matrices $D_i(s)$, find nonsingular stable polynomial matrices $X_i(s)$ (that is, the zeros of $det(X_i(s))$ lie in the left half plane $Re(s) < 0$) such that there exists a column reduced polynomial matrix $\bar{D}_{min}(s)$ such that

$$\mathcal{D}_{hc}(D_i(s)X_i(s)) = \bar{D}_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\},$$

with $\bar{D}^{hc}_{minE} = I$ and $\bar{D}_{min}(s)$ has the lowest possible degree. Here, the real matrix $\bar{D}^{hc}_{minE}$ is the highest-degree-coefficient matrix of $\bar{D}_{minE}(s)$ which is the Popov polynomial form of the matrix $\bar{D}_{min}(s)$.

In the same way that the minimal common hc-multiplier problem is related to the minimal stably based multi-realisation problem, the generic minimal common hc-multiplier problem is also related to a minimal stably based multi-realisation problem with a constraint (the minimal stably based "*generic*" multi-realisation problem). In general the degree of the solution to the minimal stably based multi-realisation problem is less than that of the minimal stably based "*generic*" multi-realisation problem. This can be seen because the specification the "*generic*" problem includes an extra condition on $\bar{D}_{min}(s)$, namely that $\bar{D}^{hc}_{minE} = I$.

Suppose that given a multi-realisation $\{A'_0 + B_0K'_{q_i}, B_0, C_{q_i}\}$ (with $\{A'_0, B_0\}$ controllable) of a particular set of systems $H_i(s) = N_i(s)D_i^{-1}(s)$, it is possible to achieve a *stably* based multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ based on the multi-realisation $\{A'_0 + B_0K'_{q_i}, B_0, C_{q_i}\}$ (without the dimension increasing, i.e. $dimA_0 = dimA'_0$ with A_0 stable). More precisely, assume we have a realisation $\{A'_0 + B_0K'_{q_i}, B_0, C_{q_i}\}$ for each $H_i(s)$ with $\{A'_0, B_0\}$ controllable by adjusting parameters K'_{qi} and C_{q_i}. If A'_0 is not stable, because $\{A'_0, B_0\}$ is controllable we can easily find a stable A_0 and a new adjustable feedback gain matrix K_{q_i} such that $A'_0 + B_0K'_{q_i} = A_0 + B_0K_{q_i}$. We may thus construct a stably based multi-realisation $\{A_0 + B_0K_{q_i}, B_0, C_{q_i}\}$ without the dimension increasing. This implies that the dimension of the minimal stably based (generic) multi-realisation form is equal to the dimension of minimal (generic) multi-realisation form. For this reason, we discuss the minimal (generic) multi-realisation problem in the following.

The hc-dependence of polynomial vectors

In order to solve Problems 4.20 and 4.22, we introduce a new concept, *hc*−(*highest column degree*) dependence on a set of polynomial vectors.

Definition 4.23 A polynomial vector $d_e(s)_{n\times 1}$ is *hc*-(highest column degree) dependent on a collection of polynomial vectors $d_i(s)_{n\times 1}$, $i = 1, 2, \cdots, m$ if there exists a set of scalar polynomials $r_i(s)$ such that

$$\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_{1}^{m} r_i(s)d_i(s)\}.$$

In Problem 4.20 and 4.22, it can be seen that each column of the minimal polynomial matrix $D_{min}(s)$ ($\bar{D}_{min}(s)$) must be *hc*-dependent on the columns of $D_i(s)$ for each $i \in \{1, 2, \cdots, N\}$.

Now, we investigate conditions for *hc*-dependence by several steps.

Lemma 4.24 (The predictable-degree property of column reduced matrices)
Let $D(s)$ be a polynomial matrix of full column rank, and for any polynomial vector $p(s)$, let

$$q(s) = D(s)p(s),$$

Then $D(s)$ is column reduced if and only if for all $p(s)$

$$deg(q(s)) = \max_{i:p_i(s)\neq 0}[deg\{p_i(s)\} + k_i]. \tag{4.33}$$

Here $p_i(s)$ is the i-th entry of $p(s)$ and k_i is the degree of the i-th column of $D(s)$.

Proof See Theorem 6.3-13 in Page 387 of [37]. ∎

Lemma 4.25 (Polynomial vector dependence over the field of rational functions) [37]

A set of polynomial vectors $d_i(s)$ ($i \in \{1, 2, \ldots, n\}$) is linearly dependent over the field of rational functions if and only if the set is linearly dependent over the field of polynomials, i.e.

$$\sum_{i=1}^{m} \alpha_i(s) d_i(s) = 0,$$

where $\alpha_i(s)$ ($i \in \{1, 2, \ldots, n\}$) are polynomial, and at least one $\alpha_i(s)$ is nonzero.

Lemma 4.26 (Non-singularity of square column reduced polynomial matrices) Let $D(s)$ be a column reduced polynomial square matrix, Then, $D(s)$ is nonsingular and all columns of $D(s)$ are linearly independent over the field of rational functions.

Proof See [37]. ∎

Lemma 4.27 Given a set of m-dimensional polynomial vectors $d_i(s)$ ($i \in \{1, 2, \ldots, m\}$), denote by d_i^{hc} the highest-(column)degree-coefficient vector of $d_i(s)$, and define $D^{hc} = [d_1^{hc} \;\; d_2^{hc}, \ldots, d_m^{hc}]$. If the real matrix D^{hc} has full column rank, then the vectors $d_i(s)$ ($i \in \{1, 2, \ldots, m\}$) are independent over the field of rational functions ($\mathcal{R}[s]$).

Proof If D^{hc} has full column rank, it is always possible to augment

$$D(s) = [d_1(s) \;\; d_2(s), \ldots, d_m(s)]$$

to be a column reduced polynomial square matrix. Then, according to Lemma 4.26, the vectors $d_i(s)$ ($i \in \{1, 2, \ldots, m\}$) are independent. ∎

Theorem 4.28 Assume there is given a collection of polynomial vectors $d_i(s)_{n\times 1}$, $i = 1, 2, \cdots, m$, such that their column degrees, k_i, are ordered as

$$k_1 \leq k_2 \leq \cdots k_m.$$

Assume further that the matrix $[d_1(s)\, d_2(s) \cdots d_m(s)]$ is column reduced. Then, a given polynomial vector $d_e(s)_{n\times 1}$ (with column degree k_e) is hc-dependent on the collection of polynomial vectors $d_i(s)$, $i = 1, 2, \cdots, m$

if and only if the real vector d_e^{hc} (the highest-(column)degree-coefficient vector of $d_e(s)$) is a linear combination of real vectors d_1^{hc}, d_2^{hc}, $\cdots$, d_l^{hc} where $l = \max_i\{\arg_i\{k_i \leq k_e\}\}$.

Proof (Forward Implication)

If

$$\mathcal{D}_{hc}\{d_e(s)\} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\},$$

for some polynomial $r_i(s)$, then

$$d_e(s) + l(s) = \sum_1^m r_i(s)d_i(s),$$

where $l(s)$ is a polynomial vector with column degree less than k_e.

According to Lemma 4.24, we have

$$d_e(s) + l(s) = \sum_1^{k_e} r_i(s)d_i(s). \tag{4.34}$$

If d_e^{hc} is not a linear combination of real vectors d_1^{hc}, d_2^{hc}, $\cdots$, d_l^{hc}, thus according to Lemma 4.27 equation (4.34) is impossible. Then, the necessity is proved.

(Reverse Implication)

If the real vector d_e^{hc} is a linear combination of real vectors d_1^{hc}, d_2^{hc}, $\cdots$, d_l^{hc}, then

$$d_e^{hc} = \Sigma_{i=1}^l r_i d_i^{hc},$$

where r_i, for $i \in \{1, 2, \cdots, l\}$ are real numbers.

It follows that

$$d_e^{hc} s^{k_e} = \Sigma_{i=1}^l r_i s^{k_e - k_i} d_i^{hc} s^{k_i}.$$

Therefore, setting $r_i(s) = r_i s^{k_e - k_i}$, we have

$$\mathcal{D}_{hc}\{d_e(s)\} = d_e^{hc} s^{k_e} = \mathcal{D}_{hc}\{\Sigma_{i=1}^l r_i s^{k_e - k_i} d_i^{hc} s^{k_i}\} = \mathcal{D}_{hc}\{\sum_1^m r_i(s)d_i(s)\}.$$

∎

A solution of the generic minimal common hc-multiplier problem

In this part, we investigate Problem 4.22, the generic minimal common hc-multiplier problem. The solution of this problem is a key to the minimal generic multi-realisation problem.

Now, we deal with the generic minimal common hc-multiplier problem in several steps

Lemma 4.29 Assume there is given a set of square $(m \times m)$ column-reduced polynomial matrices $D_i(s)$. Suppose that a polynomial matrix $\bar{D}_{min}(s)$ is a generic minimal common hc-multiplier for the set of polynomial matrices $D_i(s)$, i.e., there exist nonsingular polynomial matrices $X_i(s)$ such that

$$\mathcal{D}_{hc}(D_i(s)X_i(s)) = \bar{D}_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\},$$

with $\bar{D}^{hc}_{minE} = I$ and $\bar{D}_{min}(s)$ has the lowest possible degree. Here, the real matrix $\bar{D}^{hc}_{minE}$ is the highest-degree-coefficient matrix of $\bar{D}_{minE}(s)$ which is the Popov polynomial form of the matrix $\bar{D}_{min}(s)$.

Then, for any set of square $(m \times m)$ column-reduced polynomial matrices $\tilde{D}_i(s) = D_i(s)U_i(s)$ where $U_i(s)$ are arbitrary given unimodular polynomial matrices, there exist nonsingular polynomial matrices $\tilde{X}_i(s)$ such that

$$\mathcal{D}_{hc}(\tilde{D}_i(s)\tilde{X}_i(s)) = \bar{D}_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\},$$

and no alternative solution of Problem 4.22 for the set of polynomial matrices $\tilde{D}_i(s)$ has a lower degree than that of $\bar{D}_{min}(s)$. Equivalently, $\bar{D}_{min}(s)$ is also a generic minimal common hc-multiplier for the derived set of polynomial matrices $\tilde{D}_i(s)$.

Proof If we simply choose $\tilde{X}_i(s) = U_i^{-1}(s)X_i(s)$, then it can be seen that

$$\mathcal{D}_{hc}(\tilde{D}_i(s)\tilde{X}_i(s)) = \bar{D}_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\}.$$

We prove that matrix $\bar{D}_{min}(s)$ has the lowest possible degree by contradiction. Thus **assume** that there exists another set of nonsingular polynomial matrices $\hat{X}_i(s)$ together with a $\hat{D}_{min}(s)$ such that

$$\mathcal{D}_{hc}(\tilde{D}_i(s)\hat{X}_i(s)) = \hat{D}_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\},$$

with $\hat{D}^{hc}_{minE} = I$ and the degree of the polynomial matrix $\hat{D}_{min}(s)$ is less than the degree of $\bar{D}_{min}(s)$. Here, the real matrix $\hat{D}^{hc}_{minE}$ is the highest-degree-coefficient matrix of $\hat{D}_{minE}(s)$ which is the Popov polynomial form of the matrix $\hat{D}_{min}(s)$.

If so, we select $X_i(s) = U_i(s)\hat{X}_i(s)$; then we have

$$\mathcal{D}_{hc}(D_i(s)X_i(s)) = \hat{D}_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\}.$$

That is, the matrix $\bar{D}_{min}(s)$ is not a generic minimal common hc-multiplier for the polynomial matrices $D_i(s)$. This **contradicts** the assumption that $\bar{D}_{min}(s)$ has lowest possible degree as a solution to Problem 4.22 for the given $D_i(s)$. ∎

Lemma 4.29 indicates that the set of matrices $D_i(s)$ have the same generic minimal common hc-multiplier as the set corresponding the same set of matrices after post multiplication by a polynomial unimodular matrix.

Now, we present a method which uses elementary column operations and multiplication of columns by powers of $(s + a)$ (for some $a > 0$) to achieve a generic minimal common hc-multiplier for a set of polynomial matrices $D_i(s)$ $(i \in \{1, 2, \ldots, N\})$ (see Problem 4.22).

Method 4.30 Step 1. Transform each $D_i(s)_{m\times m}$ (by post multiplication with a unimodular matrix) to its Popov polynomial-echelon form $D_{Ei}(s)$. According to Lemma 4.29, we can seek the generic minimal common hc-multiplier for the $D_{Ei}(s)$ instead of that for the $D_i(s)$.

Step 2. By using column permutation, re-order the columns of each $D_{Ei}(s)$ to make the j_{th} column pivot index of the re-ordered matrix equal to j. Thus the ordered set of column degrees of the re-ordered matrix is equal to the ordered set of controllability indices (see Theorem 4.18). We define these indices as

$$\tilde{k}^i_1, \tilde{k}^i_2, \cdots, \tilde{k}^i_m, i \in 1, 2, \cdots, N,$$

and denote the new polynomial matrix (which is not in Popov polynomial-echelon form) as $\tilde{D}_{Ei}(s)$.

Now set

$$\begin{cases} \gamma_1 & = & \max_i\{\tilde{k}_1^i\} \\ \gamma_2 & = & \max\{\gamma_1, \tilde{k}_2^1, \tilde{k}_2^2, \cdots, \tilde{k}_2^N\} \\ \vdots & \vdots & \\ \gamma_m & = & \max\{\gamma_{m-1}, \tilde{k}_m^1, \tilde{k}_m^2, \cdots, \tilde{k}_m^N\}. \end{cases} \tag{4.35}$$

Hence, $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m$ and $\gamma_j \geq \tilde{k}_j^i$, $\forall i \in 1, 2, \cdots, N, \forall j \in 1, 2, \cdots, m$.

Step 3. Let $\Lambda_i(s) = diag\{(s+a)^{\gamma_1 - \tilde{k}_1^i}, (s+a)^{\gamma_2 - \tilde{k}_2^i}, \cdots, (s+a)^{\gamma_n - \tilde{k}_n^i}\}$ for some $a > 0$.

Define $\bar{D}_{Ei}(s) = \tilde{D}_{Ei}(s)\Lambda_i(s)$, so that $\bar{D}_{Ei}(s)$ has ordered column indices $\gamma_1 \leq \gamma_2 \leq \cdots \leq \gamma_m$. It follows that the $\bar{D}_{Ei}(s)$ are in Popov form, and according to Theorem 4.18, the highest-(column)degree-coefficient matrix for each $\bar{D}_{Ei}(s)$, $i \in \{1, 2, \cdots, N\}$ is the identity matrix.

By rewriting $H_i(s) = N_i(s)D_i^{-1}(s)$ as $N_{Ei}(s)D_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\tilde{D}_{Ei}^{-1}(s) = \tilde{N}_{Ei}(s)\Lambda_i(s)[\tilde{D}_{Ei}(s)\Lambda_i(s)]^{-1} = \bar{N}_{Ei}(s)\bar{D}_{Ei}^{-1}(s)$, it can be see that the necessary and sufficient conditions as in Theorem 4.19 for the multi-realisation of a set of multi-variable systems are satisfied, and a generic multi-realisation form $\bar{D}_m(s)$ can be achieved as $\mathcal{D}_{hc}(\bar{D}_{Ei}(s)) = diag\{s^{\gamma_1}, \cdots, s^{\gamma_m}\}$.

Method 4.30 presents a way to derive a generic common hc-multiplier of a set of square polynomial matrices. The following theorem will confirm that it is also a minimal generic common hc-multiplier.

Lemma 4.31 Denote the highest-(column)degree-coefficient vector of a polynomial vector $p(s)_{m\times 1}$ by a real vector $p_{m\times 1}^{hc}$. Suppose the elements of p^{hc} are structured as

$$p^{hc} = [p_1\, p_2\, \cdots\, p_{l-1}\, 1\, 0\, \cdots\, 0]^T, \tag{4.36}$$

and define k as the column degree of the polynomial vector $p(s)$. For a Popov polynomial matrix $D_E(s)_{m\times m}$, denote the i-th column degree by k_i, and the i-th column pivot index by p_i. Further denote the pivot index of the t_{th} column by l, i.e. $p_t = l$. If the polynomial vector $p(s)_{m\times 1}$ is hc-dependent on the columns of the Popov polynomial matrix $D_E(s)$, then

$$k \geq k_t. \tag{4.37}$$

Proof The polynomial vector $p(s)_{m\times 1}$ is hc-dependent on the columns of the Popov polynomial matrix $D_E(s)$. According to the properties of hc-dependence (see Theorem 4.28), it follows that p^{hc} is in the range of $[d_{E1}^{hc}\, d_{E2}^{hc} \cdots d_{Eq}^{hc}]$ (here q is the number of columns of the matrix $D_E(s)$ whose degree is no more than k, i.e. $q = \max_i\{\arg_i\{k_i \leq k\}\}$ $(i \in \{1, 2, \ldots, m\})$), and each of the q column degrees of $D_E(s)$ is less than or equal to k. Considering equation (4.36), we conclude the column whose pivot index is equal to l must be one of these q columns. That is, $k \geq k_t$ with $p_t = l$. ∎

Theorem 4.32 The generic common hc-multiplier $\bar{D}_m(s)$ for a set of polynomial matrices $D_{Ei}(s)$ $(i \in \{1, 2, \ldots, N\})$ (see Problem 4.22) achieved by using Method 4.30 is also a **minimal** generic common hc-multiplier.

Proof For any generic common hc-multiplier $\bar{D}_{\bar{m}}(s)$ for the set of polynomial matrices $D_{Ei}(s)$ $(i \in \{1, 2, \ldots, N\})$, we have

$$\mathcal{D}_{hc}\{D_{Ei}(s)X_i(s)\} = \bar{D}_{\bar{m}}(s), \tag{4.38}$$

and $\bar{D}_{\bar{m}}(s)$ has lowest possible degree. Denote

$$\begin{aligned} X_i(s) &= [x_{i1}(s)\, x_{i2}(s) \cdots x_{im}(s)], \\ \bar{D}_{\bar{m}}(s) &= [\bar{d}_{\bar{m}_1}(s)\, \bar{d}_{\bar{m}_2}(s) \cdots d_{\bar{m}_m}(s)]. \end{aligned} \tag{4.39}$$

From equation (4.38), we have

$$\mathcal{D}_{hc}\{D_{Ei}(s)x_{ij}(s)\} = \bar{d}_{\bar{m}_j}(s) = \mathcal{D}_{hc}\{\bar{d}_{\bar{m}_j}(s)\}.$$

That is each column of $\bar{D}_{\bar{m}}(s)$ is hc-dependent on the columns of each matrix $D_{Ei}(s)$. Note that the generic multiplier gives $\bar{D}_{\bar{m}}^{hc} = I$. Then, the j_{th} column pivot index of the matrix $\bar{D}_{\bar{m}}(s)$ is equal to j. For each matrix $D_{Ei}(s)$, denote as j the pivot index of the t_{th} column, i.e. $p_t^i = j$ (denoting the t_{th} column pivot index by p_t^i for each matrix $D_{Ei}(s)$, for $i \in \{1, \cdots, N\}$, and $t \in \{1, \cdots, m\}$). From Lemma 4.31, we conclude that for the j-th column degree $\bar{k}_{\bar{m}_j}$ of the matrix $\bar{D}_{\bar{m}}(s)$

$$\bar{k}_{\bar{m}_j} \geq k_t^i, \forall i \in \{1, 2, \cdots, N\},$$

where k_t^i is the t_{th} column degree of each matrix $D_{Ei}(s)$. By considering equation (4.27) of Theorem 4.18, we can easily see that

$$\bar{k}_{\bar{m}_j} \geq k_t^i = \tilde{k}_j^i, \forall i \in \{1, 2, \cdots, N\},$$

where $\tilde{k}_j^i$ is defined as in Method 4.30.

Further, because $\bar{D}_{\bar{m}}^{hc} = I$ implies $\bar{k}_{\bar{m}_1} \leq \bar{k}_{\bar{m}_2} \leq \cdots, \leq \bar{k}_{\bar{m}_m}$ by Theorem 4.18, we have

$$\begin{cases} \bar{k}_{\bar{m}_1} \geq \gamma_1 & = & \max_i\{\tilde{k}_1^i\} \\ \bar{k}_{\bar{m}_2} \geq \gamma_2 & = & \max\{\gamma_1, \tilde{k}_2^1, \tilde{k}_2^2, \cdots, \tilde{k}_2^N\} \\ & \vdots & \vdots \\ \bar{k}_{\bar{m}_m} \geq \gamma_m & = & \max\{\gamma_{m-1}, \tilde{k}_m^1, \tilde{k}_m^2, \cdots, \tilde{k}_m^N\}. \end{cases} \tag{4.40}$$

This means that the generic common hc-multiplier $\bar{D}_m(s)$ for a set of polynomial matrices $D_{Ei}(s)$ $(i \in \{1, 2, \ldots, N\})$ achieved by using Method 4.30 is a **minimal** generic common hc-multiplier. ∎

Note 4.33 Given the ordered controllability indices of each minimal realisation of $D_l^{-1}(s)$ $(l \in \{1, 2, \ldots, N\})$, then the minimal generic common hc-multiplier of the set of polynomial matrices $D_l(s)$ achieved by using Method 4.30 is invariant under any change of the Popov real parameters $\{\alpha_{ijk}^l\}$ for $l \in \{1, 2, \cdots, N\}$. This is a consequence of the definition of the γ_i in (4.40), and the form of $\bar{D}_m(s)$ established at the end of the Method 4.30. It is this property which justifies the use of the word "generic".

A solution of the minimal common hc-multiplier problem

In this part, we investigate Problem 4.20, the problem equivalent to the minimal multi-realisation problem. We will present some properties of a minimal common hc-multiplier for a set of polynomial matrices.

Lemma 4.34 Assume there is given a set of square $(m \times m)$ column-reduced polynomial matrices $D_i(s)$. If a polynomial matrix $D_{min}(s)$ is a minimal common hc-multiplier for the set of polynomial matrices $D_i(s)$, then, the polynomial matrix $\bar{D}_{min}(s)$ is also a minimal common hc-multiplier for a derived set of square $(m \times m)$ column-reduced polynomial matrices $\tilde{D}_i(s) = D_i(s)U_i(s)$ where $U_i(s)$ are arbitrary given unimodular polynomial matrices.

Proof The proof is analogous to the proof of Lemma 4.29. ∎

Lemma 4.35 If a column reduced polynomial matrix $D_{min}(s)$ is a minimal common hc-multiplier for a set of polynomial matrices $D_i(s)$ ($i \in \{1, 2, \ldots, N\}$) (see Problem 4.20), then, any polynomial matrix derived from the matrix $D_{min}(s)$ by column reordering is also a minimal common hc-multiplier for the set of polynomial matrices $D_i(s)$.

Proof Because the polynomial matrix $D_{min}(s)$ is a minimal common hc-multiplier for a set of polynomial matrices $D_i(s)$, there exist matrices $X_i(s)$ such that

$$\mathcal{D}_{hc}(D_i(s)X_i(s)) = D_{min}(s), \quad \forall i \in \{1, 2, \ldots, N\}, \tag{4.41}$$

and $D_{min}(s)$ has the lowest possible degree. From equation (4.41), we have

$$D_i(s)X_i(s) = D_{min}(s) + D_{min}^{lci}(s),$$

where each column degree of the matrix $D_{min}^{lc}(s)$ is lower than the corresponding column degree of the matrix $D_{min}(s)$. Then,

$$D_i(s)X_i(s)V = D_{min}(s)V + D_{min}^{lci}(s)V,$$

where V is a permutation matrix. Therefore, each column degree of the matrix $D_{min}^{lc}(s)V$ is still lower than the corresponding column degree of the matrix $D_{min}(s)V$. Denote $\tilde{X}_i(s) = X_i(s)V$. We have

$$\mathcal{D}_{hc}(D_i(s)\tilde{X}_i(s)) = D_{min}(s)V, \quad \forall i \in \{1, 2, \ldots, N\},$$

and the matrix $D_{min}(s)V$ has the same degree as the matrix $D_{min}(s)$. That is, the matrix $D_{min}(s)V$ is also a minimal common hc-multiplier for the set of polynomial matrices $D_i(s)$. ∎

Lemma 4.35 shows that after column reordering, a minimal common hc-multiplier is still a minimal common hc-multiplier for a set of polynomial matrices.

Therefore, we may assume that a minimal common hc-multiplier for a set of polynomial matrices is "column degree ordered" which along with further terminology is defined as follows [64].

Definition 4.36 A column reduced polynomial matrix $D(s)$ is said to be "column degree ordered", if the columns of the matrix $D(s)$ are ordered according to increasing column degrees $k_1 \leq k_2 \leq \cdots \leq k_m$. Suppose

$$k_1 = \cdots = k_{r_1} < k_{r_1+1} = \cdots = k_{r_1+r_2} < k_{r_1+r_2+1} \cdots \leq k_m, \qquad (4.42)$$

that is, the columns of the matrix are arranged in groups of r_i columns with the same column degree.

Let the number of groups of columns with equal degree be q, so that $k_{r_1+r_2+\cdots+r_q} = k_m$ and note that each column group has the same column degree k_i^{group} $(i \in \{1, 2, \ldots, q\})$. Further define

$$\sigma_i = \sum_{j=1}^{i} r_j, i \in \{1, 2, \ldots, q\},$$

and also define

$D_j(s)$: the sub-matrix derived from $D(s)$ by deleting the columns whose column degree are greater than k_j^{group} $(j \in \{1, 2, \ldots, q\})$.

D_j^{hc}: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_j(s)$.

Definition 4.37 A column reduced square polynomial matrix $D_u(s)_{m\times m}$ is termed hc-dependent on another square polynomial matrix $D(s)_{m\times m}$ if there exists a polynomial matrix $X(s)$ such that

$$\mathcal{D}_{hc}\{D(s)X(s)\} = \mathcal{D}_{hc}\{D_u(s)\}.$$

Theorem 4.38 Assume that a column reduced polynomial matrix $D_u(s)_{m\times m}$ is "column degree ordered" (see Definition 4.36). Define notation and symbols as following:

- k_j^{group}: the column degree of group r_j $(j \in \{1, 2, \cdots, q\})$ of the matrix $D_u(s)$.

- $D_{u_j}(s)$: the sub-matrix derived from $D_u(s)$ by deleting the columns whose column degrees are greater than k_j^{group} $(j \in \{1, 2, \ldots, q\})$.

- $D_{u_j}^{hc}$: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{u_j}(s)$.

Further assume that a Popov polynomial matrix $D_E(s)_{m\times m}$ is given. Its related notation and symbols are defined as following:

- D_E^{hc}: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_E(s)$.

- $D_{E_j}(s)$: the sub-matrix derived from $D_E(s)$ by deleting the columns whose column degree are greater than k_j^{group} ($j \in \{1, 2, \ldots, q\}$) (when all the column degrees of the matrix $D_E(s)$ are not greater than k_j^{group}, the matrix $D_{E_j}(s)$ is equal to the matrix $D_E(s)$).

- $D_{E_j}^{hc}$: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{E_j}(s)$.

Then the polynomial matrix $D_u(s)$ is hc-dependent on the Popov polynomial matrix $D_E(s)$ if and only if there exist a set of real matrices X_j with $j \in \{1, 2, \cdots, q\}$ such that

$$D_{E_j}^{hc} X_j = D_{u_j}^{hc}, \forall j \in \{1, 2, \cdots, q\}. \tag{4.43}$$

Proof By considering necessary and sufficient conditions for hc-dependence of a polynomial vector given in Theorem 4.28 and noting that $D_u(s)$ is "column degree ordered", the conclusion is straightforward. ∎

Theorem 4.38 presents a condition (see equation (4.43)) for hc-dependence of a polynomial matrix. Next, we will consider the minimal common hc-multiplier for a set of polynomial matrices based on this theorem. Specifically, we present a method which uses elementary column operations and multiplication of columns by powers of $(s + a)$ to achieve a common hc-multiplier of a set of polynomial matrices $D_i(s)$ ($i \in \{1, 2, \cdots, N\}$) (see Problem 4.20). This method consists of searching for a set k_j^{group} and σ_j ($j \in \{1, 2, \cdots, q\}$) in order to construct a common hc-multiplier $D_m(s)$

which is "column degree ordered". Later, we will prove the method provides a minimal common hc-multiplier.

We list some notation and symbols which will be used in this method. The notation is analogous to that in Theorem 4.38. Readers can skip the list of the notation and symbols unless misunderstanding occurs.

- $D_i(s)$: a set of polynomial matrices ($i \in \{1, 2, \ldots, N\}$) considered in the following method.

- $D_m(s)$: a common hc-multiplier of the set of polynomial matrices $D_i(s)$ ($i \in \{1, 2, \ldots, N\}$) achieved by using the following method.

- k_j^{group}: the column degree of group j ($j \in \{1, 2, \cdots, q\}$) of matrix $D_m(s)$.

- $D_{m_j}(s)$: a sub-matrix derived from $D_m(s)$ by deleting the columns whose column degree are greater than k_j^{group} ($j \in \{1, 2, \ldots, q\}$).

- $D_{m_j}^{hc}$: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{m_j}(s)$.

- $D_{Ei}(s)$: the Popov polynomial matrix of matrix $D_i(s)$ ($i \in \{1, 2, \ldots, N\}$).

- k_{ij}: the j_{th} column degree of the matrix $D_{Ei}(s)$($i \in \{1, 2, \ldots, N\}, j \in \{1, 2, \ldots, m\}$).

- k_1^{max}: the highest degree of the first column in all the $D_{Ei}(s)$, i.e. $k_1^{max} = \max_i\{k_{i1}\}$.

- D_{Ei}^{hc}: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{Ei}(s)$ ($i \in \{1, 2, \ldots, N\}$).

- $D_{Ei_j}(s)$: a sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degrees are greater than k_j^{group} ($j \in \{1, 2, \ldots, q\}$, $i \in \{1, 2, \ldots, N\}$).

- $D_{Ei_j}^{hc}$: the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{Ei_j}(s)$ ($j \in \{1, 2, \ldots, q\}$, $i \in \{1, 2, \ldots, N\}$).

- l_{Ei_j}: the number of columns whose degree is no more than k_j^{group} of matrix $D_{Ei}(s)$ ($j \in \{1, 2, \ldots, q\}$, $i \in \{1, 2, \ldots, N\}$), i.e. $l_{Ei_j} = \max_s\{\arg_s\{k_s \leq k_j^{group}\}\}$ ($s \in \{1, 2, \ldots, m\}$).

Method 4.39 Step 1. Transform each $D_i(s)$ to its Popov polynomial-echelon form $D_{Ei}(s)$ by post multiplication by a unimodular polynomial matrix. According to Lemma 4.34, any minimal common hc-multiplier of the set $D_{Ei}(s)$ will be a minimal common hc-multiplier of the set $D_i(s)$.

Step 2. Consider the matrices $D_{Ei}(s)$, define k_1^{max} as the highest degree of the first column in all $D_{Ei}(s)$, i.e. $k_1^{max} = \max_i\{k_{i1}\}$. By multiplication by $(s+a)^{k_1^{max}-k_{ij}}$ of any column whose column degree k_{ij} is less than k_1^{max}, one can make each $D_{Ei}(s)$ to have the lowest column degree k_1^{max}. Here k_{ij} is the j_{th} column degree of the matrix $D_{Ei}(s)$. Denote each transformed matrix as $D_{Ei}^0(s)$.

Step 3. We search for a value of k_1^{group} starting from k_1^{max}, and trying in turn k_1^{max}, $k_1^{max}+1, \cdots$ until a certain condition (given by equation (4.44) below) is satisfied. In more detail, set $k_1^{group} = k_1^{max}$ first. For each $D_{Ei}(s)$, denote $D_{Ei_1}(s)$ as the sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degree are greater than k_1^{group} ($i \in \{1, 2, \ldots, N\}$), and $D_{Ei_1}^{hc}$ as the highest-(column)degree-coefficient matrix of the polynomial matrix $D_{Ei_1}(s)$ ($i \in \{1, 2, \ldots, N\}$).

a) If for the set of real matrices $D_{Ei_1}^{hc}$, there exist constant real matrices X_{i1} and a real matrix $D_{m_1}^{hc}$, such that

$$D_{Ei_1}^{hc} X_{i1} = D_{m_1}^{hc}, \forall i, \tag{4.44}$$

with $D^{hc}_{m_1}$ has full column rank and the **largest** possible number of columns ($\sigma_1 > 0$), then it is possible to post-multiply by a real constant matrix to make each $D^0_{Ei}(s)$ (achieved in Step 2) with the same $D^{hc}_{m_1} \in \mathcal{R}m \times \sigma_1$ for the first σ_1 columns. Denote each transformed matrix as $D^1_{Ei}(s)$.

b) If for $k_1^{group} = k_1^{max}$, equation (4.44) has no solution (i.e. $\sigma_1 = 0$), then we increase k_1^{group} by one (that is $k_1^{group} = k_1^{max} + 1$). Keep on searching until a **minimal** value of k_1^{group} is achieved such that equation (4.44) has a solution X_{i1} for $i = 1, \cdots, N$. By multiplication by $(s + a)^{k_1^{group} - k_{ij}}$ of any column whose column degree k_{ij} is less than k_1^{group}, one can make each $D_{Ei}(s)$ have the lowest column degree k_1^{group}. If we denote each transformed matrix as $D^{0'}_{Ei}(s)$, then it is possible to post-multiply by a real constant matrix to make each $D^{0'}_{Ei}(s)$ have the same corresponding $D^{hc}_{m_1} \in \mathcal{R}m \times \sigma_1$ for the first σ_1 columns. Denote each transformed matrix as $D^1_{Ei}(s)$.

There always exists a value of $k_1^{group} \leq k^{max}$ (where k^{max} is the highest column degree of all $D_{Ei}(s)$, i.e. $k^{max} = max_{i,j}\{k_{ij}\}$, for $i \in \{1, \cdots, N\}$, $j \in \{1, \cdots, m\}$) such that equation (4.44) has a solution. For the case $k_1^{group} = k^{max}$, a common hc-multiplier will be $s^{k^{max}} I_m$.

That is either a) or b) will be done in this step.

Step 4. Search for a **minimal** integer k_2^{group} (searching from $k_1^{group} + 1$) and a set of real matrix X_{i2} for the set of polynomial matrices $D_{Ei}(s)$ such that

$$D^{hc}_{Ei_2} X_{i2} = D^{hc}_{m_2}, \forall i, \tag{4.45}$$

with $D^{hc}_{m_2} \in \mathcal{R}^{m \times \sigma_2}$ having full column rank and the **largest** possible number ($\sigma_2 > \sigma_1$) of columns . Recall that $D^{hc}_{Ei_2}$ is the highest-(column)degree-coefficient matrix of each $D_{Ei_2}(s)$ which is a sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degrees are greater than k_2^{group}. Define for each $D_{Ei}(s)$, $l_{Ei_2} = \max_j\{\arg_j\{k_j \leq k_2^{group}\}\}$.

Multiply the polynomial matrices $D^1_{Ei}(s)$ (achieved in Step 3) by $(s + a)^{k_2^{group} - k_{ij}}$ $(\sigma_1 < j \leq l_{Ei_2})$ from the $(\sigma_1 + 1)$-th column to the l_{Ei_2}-th column, and denote the new matrices so obtained as $D^{1'}_{Ei}(s)$. Then it is possible to post multiply by a corresponding unimodular polynomial matrix to transform each matrix $D^{1'}_{Ei}(s)$ to have the same $D^{hc}_{m_1} \in \mathcal{R}^{m \times \sigma_1}$ for the first σ_1 columns (with column degree all equal to k_1^{group}), and the same $D_{m\Delta 2}^{hc} \in \mathcal{R}^{m\times(\sigma_2-\sigma_1)}$ ($D_{m_2}^{hc} = [D_{m_1}^{hc} \vdots D_{m\Delta 2}^{hc}]$) for the columns from $(\sigma_1+1)_{th}$ columns to σ_{2th} columns (with column degree all equal to k_2^{group}).

Repeat Step 4. This will eventually derive the the common hc-multiplier for all polynomial matrices $D_i(s)$.

In this process, the values of k_i^{group} and σ_i are determined in an identical manner to those in Step 3-4. If we define $D_{Ei_j}(s)$ ($i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, q\}$) as a sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degrees are greater than k_j^{group}, and $D_{Ei_j}^{hc}$ is the highest-(column)degree-coefficient matrix of $D_{Ei_j}(s)$, then, there exist a set of real matrices X_{ij} ($i \in \{1, 2, \ldots, N\}$ and $j \in \{1, 2, \ldots, q\}$) such that

$$D_{Ei_j}^{hc} X_{ij} = D_{m_j}^{hc}, \forall i \in \{1, 2, \ldots, N\}, \forall j \in \{1, 2, \ldots, q\}. \tag{4.46}$$

The real matrix $D_{m_j}^{hc}$ has full column rank, and σ_i is equal to the number of columns in the matrix $D_{m_j}^{hc}$.

Note 4.40 1. To derive a solution for equations (4.44) and (4.45) or to identify that no such solution exists is not difficult, because each column of each D_{Ei}^{hc}, the highest-(column)degree-coefficient matrix of $D_{Ei}(s)$, has a unique pivot index.

2. In Method 4.39, if we denote $X_i(s)$ as the matrix corresponding to all column operations that were performed for each $D_{Ei}(s)$, then one can see that the zeros of $det(X_i(s))$ lie in the left half plane $Re(s) < 0$ (recall that $a > 0$ for each $(s+a)_i^k$ in Step 2-4 of Method 4.39).

3. Actually, we could skip Step 1, at least in principle. That is, we could directly perform operations on $D_i(s)$ instead of its Popov form $D_{Ei}(s)$. However, equations (4.44) and (4.45) would be more difficult to solve or identify as insolvable and the common hc-multiplier may not be a Popov polynomial form.

4. It is easy to see that the common hc-multiplier of the set of polynomial matrices $D_l(s)$ achieved by using Method 4.39 is sensitive to the variation of some Popov real parameters $\{\alpha^l_{ijd_i}\}$ for $i \in \{1, 2, \cdots, m\}$, $j \in \{1, 2, \cdots, j-1\}$ and $l \in \{1, 2, \cdots, N\}$ (see equation (4.29)).

Method 4.39 presents a way to achieve a common hc-multiplier for a set of polynomial matrices. The following theorem confirms that it is also a minimal common hc-multiplier.

Theorem 4.41 The common hc-multiplier $D_m(s)$ for a set of square column reduced polynomial matrices $D_i(s)$ ($i \in \{1, 2, \ldots, N\}$) achieved by using Method 4.39 is also a **minimal** common hc-multiplier for this set of polynomial matrices $D_i(s)$ (see Problem 4.20).

Proof Method 4.39 has already confirmed that a common hc-multiplier can be achieved with associated values k_i^{group} and σ_i. The next step of the proof is just to confirm that the common hc-multiplier form defined by k_i^{group} and σ_i is also a **minimal** common hc-multiplier.

Now, suppose that a minimal common hc-multiplier is given by $D_{min}(s)$. It is always possible to transfer $D_{min}(s)$ to $\tilde{D}_{min}(s)$ by post multiplication by a permutation matrix V such that $\tilde{D}_{min}(s)$ is "column degree ordered" with corresponding parameters $\tilde{k}_i^{group}$ and $\tilde{\sigma}_i$. According to Lemma 4.35, the polynomial matrix $\tilde{D}_{min}(s)$ is also a minimal common hc-multiplier for the set of square polynomial matrices $D_i(s)$.

We now prove the desired result by contradiction. To this end, **assume** that the common hc-multiplier $D_m(s)$ (with parameters k_i^{group} and σ_i) achieved by using Method 4.39 is not a **minimal** common hc-multiplier. Then, there should exist an integer l ($l \in \{1, 2, \cdots, m\}$) such that the l_{th} column degree k_l of the multi-realisation polynomial matrix $D_m(s)$ is bigger than the l_{th} column degree $\tilde{k}_l$ of the minimal common hc-multiplier $\tilde{D}_{min}(s)$, i.e. $\tilde{k}_l < k_l$. Without loss of generality, we assume $k_l = k_j^{group}$, so that $\sigma_{j-1} < l \leq \sigma_j$ and $\tilde{k}_l < k_j^{group}$. Similarly, we assume $\tilde{k}_l = \tilde{k}_{\tilde{j}}^{group}$, then $\tilde{\sigma}_{\tilde{j}-1} < l \leq \tilde{\sigma}_{\tilde{j}}$.

According to Theorem 4.38 (See equation (4.43)), there exists a set of real matrices $\tilde{X}_{i\tilde{j}}$ such that

$$D^{hc}_{Ei_{\tilde{j}}} \tilde{X}_{i\tilde{j}} = \tilde{D}^{hc}_{min_{\tilde{j}}}, \forall i \in \{1, 2, \ldots, N\}, and \forall \tilde{j} \in \{1, 2, \cdots, \tilde{q}\}, \tag{4.47}$$

where $\tilde{D}^{hc}_{min_{\bar{j}}}$ has full column rank $\tilde{\sigma}_{\bar{j}} \geq l$, and $D^{hc}_{Ei_{\bar{j}}}$ is the highest-(column)degree-coefficient matrix of $D_{Ei_{\bar{j}}}(s)$ which is the sub-matrix derived from $D_{Ei}(s)$ by deleting the columns whose column degree is greater than $\tilde{k}_l$.

From equation (4.46) (substituting $(j-1)$ in place of j), we have

$$D^{hc}_{Ei_{j-1}} X_{i,j-1} = D^{hc}_{m_{j-1}}, \forall i \in \{1, 2, \ldots, N\}, \forall j \in \{2, \ldots, q\}. \tag{4.48}$$

Comparing equation (4.48) with equation (4.47) (and noting that the matrix $D^{hc}_{m_{j-1}}$ has full column rank σ_{j-1}, while the matrix $\tilde{D}^{hc}_{min_{\bar{j}}}$ has full column rank $\tilde{\sigma}_{\bar{j}} \geq l$,), and also noting the fact that $\sigma_{j-1} < l$, we conclude that the column rank of the matrix $\tilde{D}^{hc}_{min_{\bar{j}}}$ is greater than that of the matrix $D^{hc}_{m_{j-1}}$. Therefore, we have $\tilde{k}_l < k^{group}_{j-1}$.

However, the integers k^{group}_j and $\tilde{k}_l$ both satisfy equation (4.46) (we see this by observing that $k^{group}_j > k^{group}_{j-1}$ and $\tilde{k}_l > k^{group}_{j-1}$). Furthermore, on consideration of Step 4 of Method 4.39, we recall that each k^{group}_j (j running from 2 to q) is the **minimal** integer (searching from $\sigma_{j-1}+1$) such that equation (4.46) can be satisfied, and hence we conclude that $k^{group}_j \leq \tilde{k}_l$. However, this **contradicts** our earlier statement that $\tilde{k}_l < k^{group}_j$.

Hence, the assumption is incorrect, and the conclusion of Theorem 4.41 holds. ∎

We will present a simple example to explain how to use Method 4.39 to achieve a minimal common hc-multiplier for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$.

Example 4.42 Using Method 4.39 to achieve a minimal common hc-multiplier for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$, where

$$D_{E1}(s) = \begin{bmatrix} 0 & 2s^2 & s^3 & 0 & 0 \\ 0 & s^2+5s & 0 & 0 & 0 \\ s+1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & s^5 \\ 0 & 0 & 0 & s^4 & 0 \end{bmatrix},$$

$$D_{E2}(s) = \begin{bmatrix} 0 & 2s^2+1 & 0 & 0 & s^5 \\ 0 & s^2 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^4 & 0 \\ 0 & 0 & s^3+s^2 & 0 & 0 \end{bmatrix}.$$

1. It is easy to see that the highest degree of the first column in the two Popov polynomial matrices is equal to 1, i.e. $k_1^{max} = 1$. So, we begin searching from $k_1^{max} = 1$, and we achieve that $k_1^{group} = 1, \sigma_1 = 1$, and

$$D_{E1_1}^{hc} \cdot X_{11} = D_{E2_1}^{hc} \cdot X_{21} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \cdot 1 = D_{m_1}^{hc}.$$

2. $k_2^{group} = 2, \sigma_2 = 2$, and

$$D_{E1_2}^{hc} \cdot X_{12} = D_{E2_2}^{hc} \cdot X_{22} = \begin{bmatrix} 0 & 2 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \cdot I_2 = D_{m_2}^{hc}.$$

3. $k_3^{group} = 4, \sigma_3 = 3$ (If we try $k_3^{group} = 3$, the maximum value of σ_3 ensuring satisfaction of (4.46) is equal to $\sigma_2 = 2$, which is not acceptable since the algorithm requires $\sigma_3 > \sigma_2$.)

$$D_{E1_3}^{hc} \cdot X_{13} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D_{m_3}^{hc},$$

and

$$D_{E2_3}^{hc} \cdot X_{23} = \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = D_{m_3}^{hc}.$$

4. $k_4^{group} = 5$, $\sigma_4 = 5$,

$$D_{E1_4}^{hc} \cdot X_{14} = \begin{bmatrix} 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} = D_{m_4}^{hc},$$

and

$$D_{E2_4}^{hc} \cdot X_{24} = \begin{bmatrix} 0 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = D_{m_4}^{hc}.$$

5. Then, we achieve the minmal common hc-multiplier $D_m(s)$ for two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$:

$$D_m(s) = \begin{bmatrix} 0 & 2s^2 & 0 & 0 & s^5 \\ 0 & s^2 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^5 & 0 \\ 0 & 0 & s^4 & 0 & 0 \end{bmatrix}.$$

Note 4.43 The generic minimal common hc-multiplier for the two Popov polynomial matrices $D_{E1}(s)$ and $D_{E2}(s)$ in Example 4.42 is $\bar{D}_m(s) = s^5 I_5$, which can be achieved by using Method 4.30. It is easy to see that the order of the minimal common hc-multiplier $D_m(s)$ is less than the order of the generic minimal common hc-multiplier $\bar{D}_m(s)$. However, if the Popov polynomial matrix $D_{E2}(s)$ is perturbed slightly, for example, as follows

$$D_{E2}(s) = \begin{bmatrix} 0 & 2.01s^2+1 & 0 & 0 & s^5 \\ 0.001s & s^2 & 0 & 0 & 0 \\ s & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^4 & 0 \\ 0 & 0 & s^3+s^2 & 0 & 0 \end{bmatrix}.$$

then the minimal common hc-multiplier will be

$$\begin{bmatrix} 0.402s^2 & 2.01s^3 & 0 & 0 & s^5 \\ 0.201s^2 & s^3 & 0 & 0 & 0 \\ s^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & s^5 & 0 \\ 0 & 0 & s^4 & 0 & 0 \end{bmatrix}.$$

We can see that the order of the minimal common hc-multiplier increases. However, the generic minimal common hc-multiplier $\bar{D}_m(s)$ (achieved by using Method 4.30) is invariant.

As mentioned in the Introduction of the thesis, a nonlinear theory, even one for single-input, single-output systems reflects more of the ideas in MIMO linear systems than in SISO linear systems. Although we only provided results of efficient multi-realisation for linear MIMO systems in this section, this is a first step towards a comprehensive theory of multi-controllers and multi-estimators for nonlinear systems.

4.2.6 Conclusion

This section aims to solve the minimal (and minimal "generic") stably based multi-realisation problems for linear MIMO systems which arise in the Multiple Model Adaptive Control. Firstly, we present the results for the multi-realisation of a number of linear SISO systems, and highlight some fundamental issues such as the relationship between feedback multi-realisation and coprime factorisation. Then, we provide the necessary and sufficient conditions for the multi-realisation of a family of linear multi-variable systems based on matrix fractional descriptions. Furthermore, we introduce the new concept of hc-dependence, and provide the necessary and sufficient conditions for hc-dependence. Finally, the minimal (and minimal "generic") stably based multi-realisation problems are solved for linear MIMO systems based on hc-dependence. All of the above results will make MMAC with its use of multi-estimator and multi-controller more efficient and practical.

Chapter 5

Conclusions

5.1 Summary

This thesis mainly concentrates on disturbance rejection for nonlinear systems based on three different methods: $\mathcal{H}_\infty$ control, singular perturbation analysis and multiple model adaptive control.

The disturbance suppression problem for nonlinear systems based on $\mathcal{H}_\infty$ control is examined in Chapter 2. This is a modest extension of nonlinear $\mathcal{H}_\infty$ theory in order to solve the constant disturbance rejection problem. We have suggested a nonlinear extension of a concept introduced for the corresponding linear problem, that of the *"comprehensively stabilising"* controller. We review the so-called nonstandard mixed sensitivity problem, which introduces an integrator to a selected weight, as well as the linear classical disturbance suppression problem and the linear $\mathcal{H}_\infty$ disturbance suppression problem. We extend this $\mathcal{H}_\infty$ problem to the nonlinear case, and present a method to reduce the order of the state feedback Hamilton-Jacobi PDE (Partial Differential Equation) for this nonlinear $\mathcal{H}_\infty$ problem by extending the concept of comprehensive stability [50] [49]. Finally, we investigate the structure of the output feedback $\mathcal{H}_\infty$ controller for disturbance suppression, and draw the conclusion that, as in the linear case, there must also be an integrator in the controller.

In Chapter 3, we have addressed the problem of achieving constant input disturbance rejection and constant reference tracking for nonlinear systems by using singular perturbation theory. A relatively practical method of suppressing the effect of constant disturbances on nonlinear systems is presented in this chapter. By adding an integrator to a stabilising controller, it is possible to achieve both constant disturbance rejection and zero tracking error. Sufficient conditions for the rejection of a constant input disturbance are given. We give both local and global conditions such that the inclusion of an integrator in the closed loop maintains closed loop stability. The analysis is based on singular perturbation theory. Furthermore, we also present some alternative locations for adding an integrator into the closed loop system and extend these methods to deal with Multiple-input Multiple-output nonlinear systems. Finally, we implement our method in the control of a simulated helicopter model. Our simulation results on the nonlinear helicopter model indicate that satisfactory performance can be achieved in some circumstances, and that the proposed method is a simple but effective way to achieve the suppression of exogenous disturbance.

The disturbance suppression problem for nonlinear systems based on Multiple Model Adaptive Control (MMAC) is examined in Chapter 4. Firstly, a stable multi-estimator for an open-loop unstable nonlinear plant is constructed based on a stable kernel representation. This is a direct extension of papers [5] and [53]. An example is presented to show the design of the multi-estimator and multi-controller to ensure constant disturbance rejection as well as constant reference tracking under plant variant. The simulation results indicate that satisfactory performance is achieved. Then, the efficient way of multi-realisation for multi-controller and multi-estimator structure, named minimal (and minimal "generic") stably based multi-realisation, is presented for linear multi-variable systems. This is an extension of paper [52], which provided the method of stably based feedback multi-realisation for linear SISO systems. Although we have not presented a comprehensive theory for multi-controllers and multi-estimators for nonlinear systems, we have constructed part of the basis of such a theory, in the consideration of MMAC for MIMO linear time invariant systems.

5.2 Further Research

As mentioned in **Chapter 2**, nonlinear $\mathcal{H}_\infty$ output feedback control is particularly difficult. The standard solution of the linear $\mathcal{H}_\infty$ output feedback control problem normally depends upon solving two Riccati equations [84]. One of these, which arises in the state feedback control problem, is replaced by an HJ PDE in the nonlinear case. The other, however, is replaced by a still more complicated equation (involving an information state), see [28]. Practical approaches to solution of this latter equation are so far lacking.

As an alternative, one can draw on ideas of nonlinear observer theory [39] [43], and substitute a state estimate $\hat{x}$ instead of the state x in a state feedback controller, retrospectively checking the γ-dissipativity and stability of the closed-loop system. In this case, the controller remains finite-dimensional, which is not normally the case when information state methods are used.

As a further research direction, a practical method for solving nonlinear output feedback $\mathcal{H}_\infty$ could be considered. There is no doubt that this is a very difficult problem. Another possible research direction is to combine the method presented in Chapter 2 with *information state methods* [28] to conquer the nonstandard nonlinear output feedback $\mathcal{H}_\infty$ control problem.

Chapter 3 demonstrates that for disturbance suppression, an output feedback controller must contain an integrator. It has presented a relatively practical way to deal with such a problem, which directly adds an integrator to an already existing controller to achieve constant disturbance rejection, while still retaining the stability of the system. However, in the chapter, there is no consideration of the robustness of the system with model uncertainty. As a next step, it would worthwhile consider the robustness with some kinds of model uncertainty. Dissipativity and passivity [1] [30] [56] [55] theory maybe a basis for the solution to this problem.

Mention was made in **Chapter 4** that paper [54] actually provided a way to achieve robust (constant) disturbance suppression and constant reference tracking for a linear SISO plant based on MMAC. The main methodology is to integrate the reference tracking error by including an integrator to the controller, a similar idea in the previous chapter. It has been shown in [54] that the controlled system can orchestrate the switching of a sequence of candidate controllers into feedback so as (i) to cause the output of the process to approach and track a constant reference input despite norm-bounded unmodelled dynamics, and constant process disturbances and (ii) to ensure that none of the signals within the overall system can grow without bound in response to bounded disturbance, be they constant or not. Many problems related to the extension of papers [5], [4], [29], [52], and [54] to the nonlinear case could be the basis for further research directions, such as the choice of nominal models, robustness analysis, securing of safe switching, boundness analysis for the disturbance - to - tracking - error gain, transient response (dwell-time switching) and so on.

Another further research task is to solve the problem of minimal stably based multi-realisation of a set nonlinear systems.

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